A RESIDUALLY FINITE ASSOCIATIVE ALGEBRA
WITH AN UNDECIDABLE WORD PROBLEM

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For any constructive commutative ring $k$ with unity, we furnish an example of a residually finite, finitely generated, recursively defined associative $k$-algebra with unity whose word problem is undecidable. This answers a question of Bokut' in [3].

Every finitely presented residually finite group has an undecidable word problem; see [1]. A similar result is also true for algebras over a constructive commutative ring $k$ if the concept 'is residually finite' is defined in terms of being approximable by $k$-algebras on finitely generated free $k$-modules. (For algebras over a field, in particular, being residually finite means being approximable by finite-dimensional algebras.) A finitely generated, recursively defined, residually finite group may have an undecidable word problem. In [3, Problem 2.10], the question is raised inquiring whether there exists a finitely generated, recursively defined, residually finite associative algebra whose word problem is undecidable. In the present article we construct a series of examples of such.

For our goal to be met, we need a construction through which an associative algebra is augmented by its endomorphism semigroup, which is similar to the well-known process via which a group is extended by its automorphism group. This construction is interesting in its own right, too.

1. AUGMENTING AN ALGEBRA BY ITS ENDOMORPHISM SEMIGROUP

Let $k$ be a commutative ring with unity, $A$ an associative $k$-algebra, and $\Phi$ a semigroup acting by endomorphisms on $A$. The latter means that an homomorphism $f$ from $\Phi$ into a semigroup of all endomorphisms of $A$ is defined. Specifically, $\Phi$ may be an arbitrary endomorphism semigroup of $A$ and $f$ can be an identity map. For $\phi \in \Phi$ and $a \in A$, denote by $a^\phi$ an image of the element $a$ under the action of an endomorphism $f(\phi)$ on $A$. In this way

$$(aa)^\phi = a a^\phi, \quad (a+b)^\phi = a^\phi + b^\phi, \quad (ab)^\phi = a^\phi b^\phi, \quad a^{\phi \psi} = (a^\phi)^\psi$$

for any $a, b \in A$, $\alpha \in k$, and $\phi, \psi \in \Phi$. Let $k \Phi$ be a semigroup algebra of the semigroup $\Phi$ over the ring $k$.

Proposition 1.1. On the $k$-module $k \Phi \otimes A$, bilinear multiplication "\cdot" can be defined uniquely so as to satisfy

$$(\phi \otimes a) \cdot (\psi \otimes b) = \phi \psi \otimes a^\psi b$$

for any $a, b \in A$ and $\phi, \psi \in \Phi$. This multiplication is associative. If $\Phi$ is equipped with unity $\varepsilon$ acting identically on $A$, and $A$ is an algebra with unity 1, then $\varepsilon \otimes 1$ is unity of multiplication.

Proof. That multiplication with the requisite property is unique is obvious, for elements of the form $\phi \otimes a$, where $\phi \in \Phi$ and $a \in A$, generate $k$ as a $k \Phi \otimes A$-module. We argue for the existence of such
multiplication. Let $b \in A$ and $s \in \Phi$. Then $s$ is of the form $\sum \beta_j \psi_j$, where $\beta_j \in k$ and $\psi_j \in \Phi$. It is easy to see that the map

$$(r, a) \mapsto \sum_j \beta_j (r \psi_j \otimes a^{\psi_j} b)$$

from $k\Phi \times A$ into $k\Phi \otimes A$ is bilinear. Therefore, there exists the homomorphism

$$f_{s,b} : k\Phi \otimes A \to k\Phi \otimes A$$

such that

$$f_{s,b}(r \otimes a) = \sum_j \beta_j (r \psi_j \otimes a^{\psi_j} b)$$

for any $r \in k\Phi$ and $a \in A$. For all $s, t \in k\Phi, b, c \in A$, and $\gamma, \delta \in k$,

$$f_{s,t+b,c} = \gamma f_{s,\delta} + \delta f_{s,t}.$$  

This follows from the fact that $r \otimes a$ generate $k\Phi \otimes A$ as a $k$-module, and from the following equalities:

$$f_{s,t+b,c}(r \otimes a) = \sum_j \gamma \beta_j (r \psi_j \otimes a^{\psi_j} b) + \sum_i \delta \beta_i (r \theta_i \otimes a^{\theta_i} b) = \gamma f_{s,b}(r \otimes a) + \delta f_{s,t}(r \otimes a),$$

and

$$f_{s+t+b,c}(r \otimes a) = \sum_j \beta_j (r \psi_j \otimes a^{\psi_j} (b+c)) = \gamma f_{s,b}(r \otimes a) + \delta f_{s,t}(r \otimes a)$$

if $s = \sum_j \beta_j \psi_j$ and $t = \sum \zeta_i \theta_i$, where $\beta_j, \zeta_i \in k$ and $\psi_j, \theta_i \in \Phi$.

Thus the map $(s, b) \mapsto f_{s,b}$ from $k\Phi \times A$ into $E(k\Phi \otimes A)$, which is a $k$-module of endomorphisms of the $k$-module $k\Phi \otimes A$, is bilinear. Therefore, there exists the homomorphism

$$g : k\Phi \otimes A \to E(k\Phi \otimes A)$$

such that $g(s \otimes b) = f_{s,b}$ for any $b \in A$ and $s \in k\Phi$. For $x \in k\Phi \otimes A$, denote $g(x)$ by $\hat{x}$; so, $\hat{x}$ is an endomorphism of the $k$-module $k\Phi \otimes A$.

For $x, y \in k\Phi \otimes A$, put $x \cdot y = \hat{x}(y)$. Obviously, this multiplication is bilinear, and for any $a, b \in A$ and $\phi, \psi \in \Phi$,

$$(\phi \otimes a) \cdot (\psi \otimes b) = f_{\psi,b}(\phi \otimes a) = \phi \psi \otimes a^{\psi} b.$$  

We verify the associativity. Let $\phi, \psi, \theta \in \Phi$ and $a, b, c \in A$. Then

$$((\phi \otimes a) \cdot (\psi \otimes b)) \cdot (\theta \otimes c) = (\phi \psi \otimes a^{\psi} b) \cdot (\theta \otimes c) = (\phi \psi) \theta \otimes (a^{\psi} b)^{\theta} c,$$

and

$$(\phi \otimes a) \cdot ((\psi \otimes b) \cdot (\theta \otimes c)) = (\phi \otimes a) \cdot (\psi \theta \otimes b^{\psi} c) = (\phi \psi \theta) \otimes a^{\psi \theta} (b^{\psi} c).$$

Since $\theta$ preserves multiplication in $A$, we have $(a^{\psi} b)^{\theta} = a^{\psi \theta} b^{\theta}$. Multiplication in $\Phi$ and in $A$ being associative implies

$$(\phi \psi) \theta \otimes (a^{\psi} b)^{\theta} c = (\phi \psi \theta) \otimes a^{\psi \theta} (b^{\psi} c).$$

The elements $\phi \otimes a$ generate a $k$-module $k\Phi \otimes A$, and so our multiplication is associative. By the same token, the last claim of the proposition is straightforward.

Denote the associative $k$-algebra introduced in Proposition 1.1 by $k\Phi \otimes A$.

**Lemma 1.2.** Let $N$ be a free $k$-module with base $X$ and $M$ be a $k$-module. Let $x_1, \ldots, x_n \in X$ be mutually distinct and $b_1, \ldots, b_n \in M$. If it is true that $\sum_j x_j \otimes b_j = 0$ in $N \otimes M$, then $b_j$ are all equal to 0.