A property of the character table for a finite group

V. A. Belonogov*

A character table $X$ of a finite group is broken up into four squares: $A$, $B$, $C$, and $D$. We establish relations via which ranks of the matrices in $X$ are connected. In particular, if $X$ is an $l \times l$-matrix, $A$ is an $s \times t$-matrix, and, moreover, the squares $A$ and $C$ are opposite, then $r(A) = r(C) + s + t - l$; here, $r(M)$ is the rank of a matrix $M$. Associated with such each block of $X$ is some integral nonnegative parameter $m$, and we have $m = 0$ iff $A$, $B$, $C$, and $D$ are active fragments of $X$.

1. INTRODUCTION

Let $G$ be a finite group, $D$ be its normal subset, and $\Phi \subseteq \text{Irr}(G)$. If $X$ is some table of characters for $G$, $X(\Phi, D)$ denotes a submatrix in $X$ lying at the intersection of rows corresponding to the characters of $\Phi$ and columns corresponding to the elements in $D$. Next, by $k(G)$ we denote the number of conjugacy classes of elements of $G$, and by $k_G(D)$ the number of such in $D$. Thus $X(\Phi, D)$ is a $|\Phi| \times k_G(D)$-submatrix of the $k(G) \times k(G)$-matrix $X$. Put $D^{-} = G \setminus D$ and $\Phi^{-} = \text{Irr}(G) \setminus \Phi$. A (suitable) character table $X$ is divided into parts as is shown in Fig. 1, where each square indicates the rank of a respective matrix. Obviously,

$$r_1 + r_2 \geq |\Phi| \quad \text{and} \quad r_1 + r_3 \geq k_G(D). \tag{1}$$

In [1], it was shown that these relations turn into equalities iff $D$ interacts with $\Phi$; see also Lemma 1 below, followed by the definition of the notion of interaction.

In the present article we study relations among $r_1$, $r_2$, $r_3$, and $r_4$ for arbitrary $D$ and $\Phi$. In particular, we give some data on differences of the left- and right-hand sides of the inequalities in (1).

The rank and the determinant of a matrix $M$ are denoted by $r(M)$ and $\det(M)$, respectively.

The key result of the article is the following:

THEOREM. Let $G$ be a finite group, $X$ its character table, $D$ a normal subset in $G$, and $\Phi \subseteq \text{Irr}(G)$. Then there exists a nonnegative integer $m$ which depends on $G$, $D$, and $\Phi$ and is such that

$$r(X(\Phi, D)) + r(X(\Phi, D^{-})) = |\Phi| + m,$$

$$r(X(\Phi^{-}, D)) + r(X(\Phi^{-}, D^{-})) = |\Phi^{-}| + m,$$

$$r(X(\Phi, D)) + r(X(\Phi^{-}, D)) = k_G(D) + m,$$

$$r(X(\Phi, D^{-})) + r(X(\Phi^{-}, D^{-})) = k_G(D^{-}) + m.$$
Moreover, \( m = 0 \) iff \( D \) interacts with \( \Phi \).

In particular, if \( G, X, D, \) and \( \Phi \) satisfy the conditions of the theorem, then

\[
r(X(\Phi, D)) = r(X(\Phi^{-}, D^{-})) + |\Phi| + k_G(D) - k(G) + 14,
\]

In turn the theorem immediately implies the result given immediately below, which states the existence of nonzero submatrices, in particular, nonzero elements, in the character table: we need only consider the difference of the first and fourth equalities in our theorem; see also [1, Thm. 8A8].

**COROLLARY.** Let \( G, X, D, \) and \( \Phi \) satisfy the conditions of the theorem. Then the following are equivalent:

1. \( r(X(\Phi, D)) = |\Phi| + k_G(D) - k(G) \);
2. \( X(\Phi^{-}, D^{-}) = 0 \) (zero matrix).

Notice: In our theorem (and its corollary), we do not presume that the sets \( D, D^{-}, \Phi, \) and \( \Phi^{-} \) are nonempty, that is, some squares of \( X \) in Fig. 1 can well be the empty matrices, by which are meant ones with 0 rows and 0 columns. The rank of the empty matrix is, as usual, assumed equal to zero.

2. **PROOF OF THE THEOREM**

We start with some auxiliary statements.

**LEMMA 1.** Let \( G, X, D, \) and \( \Phi \) be as in the theorem. Then the following conditions are equivalent:

1. \( r(X(\Phi, D)) + r(X(\Phi^{-}, D^{-})) = |\Phi| \);
2. \( r(X(\Phi, D)) + r(X(\Phi^{-}, D)) = k_G(D) \);
3. \( D \) interacts with \( \Phi \).

We can see that this is in fact part of Theorem 8A6 in [1].

Recall the definition of the notion of interaction. \( D \) is said to *interact* with \( \Phi \) if the \( D \)-cutting \( \varphi \mid_D \) of any character \( \varphi \) in \( \Phi \) is a linear combination (with complex coefficients) of the characters in \( \Phi \); \( \varphi \mid_D \) coincides with \( \varphi \) on \( D \) and vanishes on \( G \setminus D \). If \( D \) interacts with \( \Phi \), \( X(\Phi, D) \) is called an **active fragment** of the table \( X \). (A classical example where the sets \( D \) and \( \Phi \) are interacting is the case where \( D \) is the union of an arbitrary set of \( p \)-sections of \( G \) and \( \Phi \) is the union of an arbitrary set of \( p \)-blocks in \( G \), \( p \) is a prime; see [1, Sec. 5H].)