This paper deals with the problems of identifying oscillations with regular and chaotic attractors in deterministic oscillating systems. It discusses a new approach based on the dynamical principle of symmetry, the construction of aperiodic solution domains, and the analysis of characteristic indices of quasistatic solutions for circular trajectories (in polar coordinates). The Duffing equation is examined within the framework of the dynamical principle of symmetry.

One model which is widely used to study oscillatory processes is the Duffing equation. Random motions develop when systems of this type are subjected to periodic external disturbances (see [8, 13]). Wood and other authors [8] have studied random motions in electrical networks. In the recent monograph [11], the Duffing equation was used to describe processes in the dynamics of rotors with a variable mass, and the conditions under which random motions arise were found. Irregular motions also occur in problems of structural mechanics [2]. The Duffing equation can be used to model the vibrations of an elastic arch subjected to a periodic external disturbance. Studies of the behavior of the solution of this equation can also be found in [10, 14, 15]. The goal of this investigation is to establish the relationship between the symmetry of the quasistatic solution and the regularity of motions induced by a periodic disturbance.

1. Preliminary Remarks. The characteristic indices of the solutions are associated with rigorous symmetry criteria. Solutions constructed on the basis of the asymptotic theory of N. N. Bogolyubov, Yu. A. Mitropol’skii, and A. M. Samoilenko [3, 7] can be used for analysis. Then the question of the existence of an invariant toroidal manifold (for example) can be solved by means of higher approximations. The approach used in [6, 9] was based on an analysis of the quasistatic solution that enters into the approximate asymptotic solution, and whose character of the symmetry can be used to evaluate the stability and regularity of the trajectory. We will use equations in the variables \( \rho \) and \( \theta \). Asymptotic principles have also been developed for such equations [5], in which \( \rho \) is the modulus of the complex variable. Then the interaction may result in disappearance of the imaginary component from the equations of motion and appearance of the equality \( d\theta/dt = 0 \) in the equation in the variable \( \theta \).

The authors of [9] examined the Duffing equation

\[
d x^2/d\xi^2 + h dx/d\xi - x (\kappa - \gamma x^2) = f(\Omega \xi),
\]

where \( \Omega \) is the frequency of the periodic disturbance. Here, the independent variable \( \xi \) plays the role of time in problems on the vibration of systems subjected to a periodic force [1, 2]. Another type of problem concerns statically deformed elastic systems that are reduced to equations in Cauchy form [4, 9]. The variable \( \xi \) plays the role of the space coordinate. When \( h = 0 \), \( f(\Omega \xi) = 0 \), \( \kappa = \gamma = 1/2 \), Eq. (1.1) appears as follows in Cauchy form:

\[
dx_1/d\xi = x_2/2 (1 - \xi^2), \quad dx_2/d\xi = x_1.
\]

It was shown in [9] that closed orbits of oscillations with a frequency close to 1 will be formed relative to the following points in independent system (1.2), depending on the initial conditions for the trajectory:

1) a point of the center type \( x_1 = 0, x_2 = 1 \) in the case of initial conditions such as \( x_{10} = 0, 0 < x_{20} < 2 \);
2) a point of the center type $x_1 = 0, x_2 = -1$ in the case of initial conditions such as $x_{10} = 0, -2 < x_{20} < 0$;
3) the saddle point $x_1 = 0, x_2 = 0$ in the case of initial conditions such as $x_{10} = 0, -2 < x_{20} < 2$.

The closure of the trajectory is due to symmetry. Periodic disturbance of the orbit by trajectories closed around the centers $x_1 = 0, x_2 = 1$ and $x_1 = 0, x_2 = -1$ leads to random motion. The breakup of a closed trajectory occurs because of disturbance of the symmetry of the quasistatic solutions of the dependent system for single-frequency motions or motions that are nearly synchronous. A trajectory that is locally unstable enters the region of aperiodic motions. The periodic trajectory splits into two aperiodic trajectories. For trajectories that close around centers, the characteristic indices of the aperiodic motions have the form $\lambda_1^* < 0, \lambda_2^* > 0$. The stability of the chaos is determined by the upper-bounded region of aperiodic motions and the stability of the periodic motions. Our goal here is to establish the relationship between symmetry and the regularity of motions generated by periodic excitations. Some results were presented in [9]: trajectories that are closed around centers $x_1 = 0, x_2 = 1$ and $x_1 = 0, x_2 = -1$ cannot be transformed into regular trajectories by periodic excitation. Using the findings in [9], we pose the more general question: in what cases does periodic excitation generate regular motion and can the quality of that motion be predicted?

2. Regular and Irregular Trajectories. We will examine a closed plane trajectory which can be formed in a conservative nonlinear system. The trajectory is referred to polar coordinates

$$
\frac{d\rho}{d\xi} = R(\rho, \theta),
$$

$$
\frac{d\theta}{d\xi} = \Omega + T(\rho, \theta),
$$

(2.1)

where $\rho$ is the modulus of the complex variable; $\theta$ is the angular coordinate; $\Omega$ is a constant; $R(\rho, \theta)$ is a polynomial of degree no greater than three in $\rho$. Solving the equation

$$
R(\rho, \theta) = 0
$$

for $\rho$, we will refer to the resulting solution for $\theta \in (0, 2\pi)$ as quasistatic

$$
\rho = \rho^0(\theta), \quad \rho^0(\theta) \geq 0.
$$

The condition of differentiability of the right side of the equations in Cauchy form ensures that the quasistatic solution will be smooth. The smooth quasistatic solution can be discontinuous. The trajectory at each point can be attracted or repulsed relative to the solution $\rho^0(\theta)$, depending on the sign of the characteristic index

$$
\lambda(\theta) = \left. \frac{\partial R(\rho, \theta)}{\partial \rho} \right|_{\rho = \rho^0(\theta)}.
$$

We will impose one other condition on system (2.1): the derivative $d\theta/d\xi$ for the quasistatic solution $\rho^0(\theta)$ is equal to a constant in $\theta \in (0, 2\pi)$:

$$
\frac{d\theta}{d\xi}(\rho^0(\theta), \theta) = \Omega, \quad \theta \in (0, 2\pi).
$$

(2.2)

Then the following symmetry is observed in system (2.1). At equal subintervals over $\theta$, the closing trajectory is attracted to zero and repulsed from zero at the same velocity $d\rho/d\xi$; the quasistatic solution $\rho^0(\theta)$ of system (2.1) also possesses symmetry in terms of attraction and repulsion.

The regularity and stability of the trajectory of the conservative system are preserved during periodic excitation if attractive and repulsive symmetry is preserved in the evolutionary equation of the system

$$
\frac{d\tilde{\rho}}{d\xi} = R(\tilde{\rho}, \theta) + P(\theta, \Omega \xi),
$$

$$
\frac{d\theta}{d\xi} = \Omega + T(\tilde{\rho}, \theta) + Q(\tilde{\rho}, \theta, \Omega \xi).
$$

(2.3)