SOLVABLE ELIMINATION OF RAMIFICATION IN EXTENSIONS OF DISCRETELY VALUED FIELDS

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An analog of the Abhyankar lemma is proved for an arbitrary finite separable extension of discretely valued fields with a functional residue field.

INTRODUCTION

In this article we work with ramification theory of valued fields. Recall some well-known facts from that theory; see [1, Chs. 1 and 2; 2, Ch. 4; 3, Ch. 5]. If $K$ is a field and $L$ an algebraic extension, then every valuation of $K$ extends to a valuation of $L$. We are interested in the question of what properties are shared by such extensions.

If $K$ is a valued field, $k(K)$ denotes a residue field of a valuation ring of $K$. Let $L/K$ be the field extension with conforming valuations. That extension determines a residue field extension $k(L)/k(K)$. If $L/K$ is finite, then so is $k(L)/k(K)$ of course. A degree of extension of a residue field is denoted by $f(K)$, and its separable and inseparable degrees — by $f_s(L/K)$ and $f_u(L/K)$, respectively. The properties of degrees of field extensions imply the identity $f(L/K) = f_s(L/K) + f_u(L/K)$, and these functions are multiplicative. Other multiplicative functions of extensions are defined as follows.

The value group $\Gamma(K)$ of a valuation of the field $K$ is identified with a subgroup of a value group of the extension $L$, $\Gamma(K) \leq \Gamma(L)$. An index of the subgroup is called a reduced ramification index of $L/K$, and we denote it by $e_r(L/K) = [\Gamma(L) : \Gamma(K)]$. In valued field theory, that index is called merely a ramification index; see [1, 2]. In algebraic geometry, however, a ramification index of $L/K$ is normally meant to apply to a product of a reduced ramification index and an inseparable degree of extension of a residue field: $e(L/K) = e_r(L/K) \cdot f_u(L/K)$; see [3, Ch. 5]. We make this convention throughout.

Distinguish certain classes of separable extensions of valued fields. A separable extension of valued fields is called unramified if its ramification index is trivial, $e_r(L/K) = 1$. This is equivalent to saying that the residue field extension $k(L)/k(K)$ is separable and the reduced ramification index trivial.

A broad class is formed by extensions in which only the condition on triviality of a reduced ramification index is satisfied, $e_r = 1, \Gamma(L) = \Gamma(K)$, and a residue field extension may or may not be separable. Such are called weakly unramified.

Another broad class is built from weakly ramified extensions. Let $p$ be a characteristic exponent of the field $K$. An extension of valued fields is called weakly ramified (or tame) if its ramification index $e(L/K)$ is prime to $p$, $(e(L/K), p) = 1$. This is equivalent to saying that the residue field extension $k(L)/k(K)$ is separable, and that the reduced ramification index $e_r(L/K)$ is prime to the field characteristic $p$, $(e_r(L/K), p) = 1$. But if a ramification index of an extension is a power of the field characteristic (possibly

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the extension is called **wildly ramified** (or wild). Tame and, simultaneously, wild extensions are exactly the what we call unramified extensions.

If a valued field $K$ has characteristic 0, all the extensions are tame by definition. In the nonzero case where $p \neq 0$, there may exist sufficiently many extensions that are not tame.

An extension of discretely valued fields is called pure if a degree of extension is equal to a ramification index. An extension is said to be **totally ramified** if a degree of extension coincides with a reduced ramification index.

Let $K$ be a valued field and $L/K$ some algebraic extension of $K$ without a valuation. An extension $L/K$ is called unramified (resp., weakly unramified, tame, wild, or totally ramified) if every extension of a valuation of the field $K$ to a field $L$ determines an unramified (resp., weakly unramified, tame, wild, or totally ramified) extension of valued fields.

By the known relation on reduced ramification indices and degrees of extension of residue fields of different extensions of a valuation (see [5, Ch. 12, Cor. 2 to Prop. 18]), it follows that, for a totally ramified extension $L/K$, there exists a unique extension of a valuation of $K$ to $L$. Therefore, for the case of totally ramified extensions, the above definition reduces to verifying whether some one extension of the valuation of $K$ to $L$ is totally ramified.

If $L/K$ is a Galois extension, then all extensions of the valuation of $K$ to $L$ are conjugate by automorphisms in the Galois group of the extension. For conjugate valuations, ramification indices are equal, and the automorphism induces an isomorphism of residue fields of those valuations. For the present case, therefore, the definition of a ramification type needs only be verified for some one extension of the valuation of $K$ to $L$ is totally ramified.

We recall that a separable extension is called cyclic (solvable) if the Galois group of its normal closure is cyclic (resp., solvable). This is equivalent to saying that the extension is embedded in a tower of cyclic extensions. A solvable extension is called strongly solvable if it is itself presented by a tower of cyclic extensions. But if all stages of the tower are cyclic extensions of the same degree $p$, which is a prime, the initial extension is referred to as a $p$-extension. In algebraic geometry, the ramification theory of valued fields serves as a tool for studying coverings of algebraic varieties. In studies of such coverings with tame ramification, the fact — known under the name of Abhyankar's lemma — occupies center-stage; see [7, Sec. 10, Lemma 3.6, p. 19]. That result was obtained by Abhyankar in the mid-50s, in a series of articles dealing with coverings of algebraic varieties, and can be worded thus. Let $K$ be a discretely valued field of arbitrary characteristic containing all roots of unity, and let $L$ be its finite extension. Suppose that $L/K$ is tame. In the algebraic closure of $K$, then, there exists a cyclic totally ramified extension $K'$ of $K$ over which $L \cdot K'$ is unramified, for any extension of the valuation of $K$ to $K'$. The extension $K'/K$ proper is tame relative to every extended valuation.

An attempt to generalize Abhyankar's lemma and remove from it the requirement for $L/K$ to be weakly ramified was undertaken in [8]. In a language of discretely valued fields, that result is worded thus. Let $L/K$ be a (not necessarily finite and algebraic) extension of discretely valued fields, with some necessary conditions imposed on a residue field extension $k(L)/k(K)$. In some larger field, then, we build up a finitely generated extension $K'$ of $K$ for which a composite $L \cdot K'$ over $K'$ is weakly unramified, for any extension of the valuation of $K$ to $K'$. That proof, however, has some gaps.*

In the present article, our treatment will be confined to finite separable extensions of a discretely valued field of nonzero characteristic $p$. In order to give a formulation of the result, we need the following:

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