ON THE ZETA FUNCTION OF A HYPERSURFACE (1)

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This article is concerned with the further development of the methods of p-adic analysis used in an earlier article [1] to study the zeta function of an algebraic variety defined over a finite field. These methods are applied to the zeta function of a non-singular hypersurface \( \mathcal{S} \) of degree \( d \) in projective \( n \)-space of characteristic \( p \) defined over the field of \( q \) elements. According to the conjectures of Weil [3] the zeta function of \( \mathcal{S} \) is of the form

\[
\zeta(\mathcal{S}, t) = P(t)(-1)^n \prod_{i=0}^{n-1} (1 - q^i t)
\]

where \( P \) is a polynomial of degree \( d^{-1}\{(d-1)^n + (-1)^n + (d-1)^{n-1}\} \), (here \( n \geq 0, d \geq 1 \), for a discussion of the trivial cases \( n = 0, 1 \) see § 4 b below). It is well known that this is the case for plane curves and for special hypersurfaces, [3]. We verify (Theorem 4.4 and Corollary) this part of the Weil conjecture provided \( 1 = (2, p, d) \), that is provided either \( p \) or \( d \) is odd.

In our theory the natural object is not the hypersurface alone, but rather the hypersurface together with a given choice of coordinate axes \( X_1, X_2, \ldots, X_{n+1} \). If for each (non-empty) subset, \( A \), of the set \( S = \{1, 2, \ldots, n+1\} \) we let \( \mathcal{S}_A \) be the hypersurface (in lower dimension if \( A \neq S \)) obtained by intersecting \( \mathcal{S} \) with the hyperplanes \( \{X_i = 0\}_{i \in A} \), then writing equation (1) for \( \mathcal{S}_A \), we define a rational function \( P_A \) by setting

\[
\zeta(\mathcal{S}_A, t) = P_A(t)(-1)^{m(A)} \prod_{i=0}^{m(A)-1} (1 - q^i t),
\]

where \( 1 + m(A) \) is the number of elements in \( A \). If \( \mathcal{S}_A \) is non-singular for each subset \( A \) of \( S \) and if the Weil conjectures were known to be true then we could conclude that \( P_A \) is a polynomial for each subset \( A \).

Our investigation rests upon the fact that without any hypothesis of non-singularity we have

\[
\chi_f^{n+1}(t) = (1 - t) \prod_A P_A(gt),
\]

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the product on the right being over all subsets \( A \) of \( S \) and \( \chi_F \) is the characteristic series
of the infinite matrix \([z]\) associated with the transformation \( \alpha = \psi_0 F \) introduced in our
previous article [1] and studied in some detail in § 2 below. We recall that
\[ \chi_F(t) = \chi_F(t) / \chi_F(qt) \]
and the fundamental fact in our proof of the rationality of the zeta
function is that \( \chi_F \) is an entire function on \( \Omega \), the completion of the algebraic closure
of \( \mathcal{O}' \), the field of rational \( p \)-adic numbers.

In § 2 we develop the spectral theory of the transformation \( \alpha \) and show that the
zeros of \( \chi_F \) can be explained in terms of primary subspaces precisely as in the theory
of endomorphisms of finite dimensional vector spaces. In this theory it is natural to
restrict our attention to a certain class of subspaces \( L(b) \) (indexed by real numbers, \( b \))
of the ring of power series in several variables with coefficients in \( \Omega \). The
definition of \( L(b) \) is given in § 2, for the present we need only mention that if \( b' > b \),
then \( L(b') \subseteq L(b) \).

An examination of (4.33) shows that if the right side is a polynomial and if \( \theta^{-1} \)
is a zero of that polynomial of multiplicity \( m \) then \( (\theta q)^{-1} \) must be a zero of \( \chi_F \) of
multiplicity \( m({n+i}) \). This is « explained » by the existence of differential operators
\( D_1, \ldots, D_{n+1} \) satisfying
\[ (4.35) \quad \alpha D_i = qD_i \alpha \]

The space \( L(b) / \bigoplus_{i=1}^{n+1} D_i L(b) \) is studied in § 3 d (in a slightly broader setting than
required for the geometric application), for \( 1/(p-1) \leq b \leq p/(p-1) \), the main results
being Lemmas 3.6, 3.10, 3.11. This is applied in § 4 to show that if \( \delta \) is non-singular
for each subset \( A \) of \( S \) then the right side of (4.33) is a polynomial of predicted degree
and is the characteristic polynomial of \( \alpha \), the endomorphism of \( L(b) / \bigoplus D_i L(b) \) obtained
from \( \alpha \) by passage to quotients. (Theorems 4.1, 4.2, 4.3) We emphasize that this
result is valid for all \( p \) (including \( p = 2 \)).

The main complication in our theory lies in the demonstration (Theorem 4.4 and
corollary) that if \( i = (2, p, d) \) then \( P_i (tq) \) is the characteristic polynomial of \( \alpha^b \),
the restriction of \( \alpha \) to the subspace of \( L(b) / \bigoplus D_i L(b) \) consisting of the image of \( L^b(b) \) under
the natural map, \( L^b(b) \) being the set of all power series in \( L(b) \) which are divisible
by \( x_n x_{n-1} \ldots x_1 \). This result is of course based on the study (§ 3 e) of the action
of the differential operators on \( L^b(b) \). This study is straightforward for \( p \nmid d \) but for \( p \mid d \)
the main results are shown to be valid only for special differential operators.

We must now explain that for a particular hypersurface we have many choices
for the operator \( \alpha \) (see § 4 a below) but once \( \alpha \) is chosen the differential operators
satisfying (4.35) are fixed. With a simple choice of \( \alpha \) the eigenvector spaces lies
in \( L \left( \frac{b-1}{p} \right) \) while for a more complicated choice of \( \alpha \) the eigenvector space is known
to lie in \( L \left( \frac{p}{b-1} \right) \). The special differential operators referred to above in connection
with the case \( p \mid d \) are those which correspond to the simple choice of \( \alpha \) for which the