On a Certain Duality for Segmented Minmax Problems

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Abstract

Minmax problems are considered, where the set of parameters consists of a finite number of knots from a real interval. A theory of a nonstandard duality is developed to some extent. Several applications as well as some computational aspects are discussed.

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1. Introduction, Preliminaries

Univariate approximation problems, in which knots occur, are of particular interest in numerical analysis. The segmented approximation by spline functions with free knots leads to well known questions, not all of which are answered satisfactorily yet. Our investigations throw a new light into the scene, since many of such problems can be interpreted as duality problems. In convex optimization, duality is said to hold, if, for given sets $X$ and $Y$ and a function $f: X \times Y \rightarrow \mathbb{R}$ the relation

$$
\inf_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \inf_{x \in X} f(x, y)
$$

holds true. Here we consider instead a nonstandard duality of the form

$$
\inf_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{x \in X} \inf_{y \in Y} f(x, y).
$$

We fix a closed interval $[a, b] \subseteq \mathbb{R}$ and an integer $k \geq 2$.

Definition. A $k$-partition $\tau$ of the interval $[a, b]$ is a real $(k + 1)$-tuplet

$$
\tau = (\tau_0, \tau_1, \ldots, \tau_k) \quad \text{with} \quad a = \tau_0 \leq \tau_1 \leq \ldots \leq \tau_k = b.
$$
The interval \([a,b]\) is segmented by a \(k\)-partition \(\tau\) into the \(k\) intervals \(I_\nu = [\tau_{\nu-1}, \tau_\nu]\) for \(\nu = 1, 2, \ldots, k\). We denote by \(\mathcal{F}_k\) the set of all \(k\)-partitions \(\tau\) of \([a,b]\).

Define the triangle \(M \subseteq \mathbb{R}^2\) as
\[
M := \{(\xi, \eta) \in \mathbb{R}^2 | a \leq \xi \leq \eta \leq b\}.
\]

Let there be given a function \(r: M \to \mathbb{R}\) having the following properties:

\begin{enumerate}
\item[(a)] \(r(\cdot, \cdot)\) is continuous on \(M\); \hfill (1)
\item[(b)] either, for all \((\xi, \eta), (\tilde{\xi}, \tilde{\eta}) \in M\),
\[
[\xi, \eta] \subseteq [\tilde{\xi}, \tilde{\eta}] \implies r(\xi, \eta) \leq r(\tilde{\xi}, \tilde{\eta}), \hfill (2)
\]
or, for all \((\xi, \eta), (\tilde{\xi}, \tilde{\eta}) \in M\),
\[
[\xi, \eta] \subseteq [\tilde{\xi}, \tilde{\eta}] \implies r(\xi, \eta) \geq r(\tilde{\xi}, \tilde{\eta}). \hfill (3)
\]
\end{enumerate}

In contrast to previous approaches we do not assume here that \(r(\xi, \xi) = 0\) for all \(\xi \in [a,b]\). This opens the way to a segmented treatment of general minmax problems, besides problems of best approximation.

**Remark 1.** The inclusion \([\xi, \eta] \subseteq [\tilde{\xi}, \tilde{\eta}]\) defines a semi-ordering on \(M\), because it is trivially equivalent to \(\tilde{\xi} \leq \xi, \eta \leq \tilde{\eta}\). The conditions (2), (3) respectively express the fact that the function \(r\) represents a monotone resp. antitone operator with respect to this semi-ordering.

**Remark 2.** Let \(r(x, y)\) possess partial derivatives on \(M\) of first order with respect to \(x\) and \(y\). Then it is easy to see that condition (2) is equivalent to the two inequalities
\[
\frac{\partial r}{\partial x} \leq 0 \quad \text{and} \quad \frac{\partial r}{\partial y} \geq 0 \quad (4)
\]
on \(M\). For (3) one gets the equivalent condition
\[
\frac{\partial r}{\partial x} \geq 0 \quad \text{and} \quad \frac{\partial r}{\partial y} \leq 0.
\]

In what follows we will always use the monotonicity condition (2). The proofs, using condition (3), are analogous.

### 2. The Duality Problem

Now for \(\tau \in \mathcal{F}_k\) and all \(\nu \in \{1, \ldots, k\}\) we let
\[
\varphi_\nu(\tau) := r(\tau_{\nu-1}, \tau_\nu).
\]