On the Feedback Vertex Set Problem for a Planar Graph

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Abstract — Zusammenfassung

On the Feedback Vertex Set Problem for a Planar Graph. An algorithm solving the feedback-vertex-set problem for planar digraphs is described. In particular, planar graphs with a certain additional condition are considered as they arise from solving systems of linear equations obtained from convection-dominated flow problems. The proposed algorithm requires a computational work linear in the size of the graph. Furthermore, a side-product is a decomposition of the graph into subsets that are of interest for block-Gauß-Seidel smoothers in multi-grid methods.

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1. Background of the Problem

1.1. Graph of a Matrix

Before posing the problem, we recall the definition of the graph of a matrix (cf. Section 6.2 in [5]). We consider a linear system \( Ax = b \) with matrix entries \( a_{PQ} \) and unknowns \( x_p \), where \( P \) is in the index set \( V \). The matrix graph \( G = G(A) \) consists of the vertex set \( V = V(G) \) and the edges \( E = E(G) = \{(P, Q) \in V \times V : a_{PQ} \neq 0\} \). All graphs used in the following are directed graphs.

Let the matrix \( A \) arise from a discretisation of a partial differential equation

\[ -\Delta u + c_1 u_x + c_2 u_y = f \quad \text{in} \ \Omega \subset \mathbb{R}^2. \quad (1.1) \]

with suitable boundary conditions by means of a finite element or difference method (cf. [4]). Often, the vertices of \( V(G) \) correspond to the vertices of the (finite) elements, while the edges in \( G \) are the edges of the elements. Corre-
spondingly, discretisations of problems in two spatial dimensions lead to a planar graph $G$.

Because of its symmetry, the diffusion term $-\Delta u$ yields symmetric edges (i.e., edges in both directions), whereas the convection term $c_1 u_x + c_2 u_y$ generates edges in only one direction.

### 1.2. Strong Convection

As long as the convection is moderate (i.e., $c_1$ and $c_2$ of size $O(1)$), multi-grid methods with simple smoothing iterations are successful (cf. [3]). When, however, convection becomes dominant, the convergence may become worse and good convergence rates require a certain ordering of unknowns. Several proposals for "robust smoothers" have been made (cf. Section 10.4 in [3]). We now follow another line: A simple Gauß-Seidel method leads to excellent results if the ordering of the unknowns follows the convection direction (the flow direction $(c_1, c_2) \in \mathbb{R}^2$).

The ideal case would be that the unknowns can be ordered so that $x_i$ depends only on $x_j$ with $j < i$. In that case, $A$ forms a lower triangular matrix and one Gauß-Seidel step is an exact solver. This ideal situation corresponds to an acyclic graph $G$. Here, we use the following notation. The tuple $(P, Q, R, \ldots, Y, Z)$ is a path in $G$, if $P, \ldots, Z \in V(G)$ and $(P, Q), (Q, R), \ldots, (Y, Z) \in E(G)$. The path is a cycle, if $P = Z$. $G$ is acyclic, if no cycles exist in $G$.

The matrix $A$ of the discrete version of problem (1.1) has a natural splitting into $A^D + A^C$, where the terms correspond to the diffusion and convection part discussed above. Since the multi-grid method works well without or with moderate convection, one has not to care about $A^D$ and it suffices to consider the graph of $A^C$. In order to suppress small convection, one can define the graph of dominant convection by edges $(P, Q) \in V \times V$ with

$$|a^C_{PQ}| > \kappa |a^D_{PQ}|,$$

where $\kappa$ may depend on the mesh size. It is also reasonable to consider only the strongest connections:

$$(P, Q) \in E(G) \leftrightarrow (1.2a) \text{ holds and } |a^C_{PQ}| \geq \frac{1}{2} \max \{|a^C_{PR}| : |a^C_{PR}| > \kappa |a^D_{PR}|\}. \quad (1.2b)$$

This defines the graph $G$ that we discuss in the following. If $G$ is acyclic, the ordering mentioned above (and constructed in Section 2.1) does not lead to an exact solver, but the multi-grid method with Gauß-Seidel smoothing would work very fast.

If $G$ has cycles, two remedies can be used. First we can try to remove some vertices such that the remaining graph becomes acyclic. Such a set of vertices is called a feedback vertex set. These vertices could form the set of indices