SEMICONTDINUOUS UTILITY FUNCTIONS IN TOPOLOGICAL SPACES

ROMANO ISLER

Dipartimento di Matematica Applicata "Bruno de Finetti"
Università di Trieste

Versione definitiva pervenuta il 11/01/99

Rader [7] in 1963 gave a proof of a theorem about the existence of an upper semi-continuous function on a second countable topological space $X$ with a total preorder $\preceq$, provided that the left half lines $\{x : x \prec y\}$ are open for each $y$ belonging to $X$. The proof of Rader was constructive and direct, without any use of theorems like the Continuum Characterization Theorem. Unfortunately the proof was wrong, as proved by Mehta [6] in 1997. Following the Mehta's remark and using in a "good way" the axioms, we give a new constructive and direct proof which is presented in 2.3. Indeed there is also another proof of Rader's theorem due to M. K. Richter [8]. The proof is based on a sufficient condition for a function to be upper-semicontinuous, less direct and general as ours.

1. Binary relations

1.1. Let $X$ be a set and $\prec$ a binary relation. We will use the following axioms for $\prec$:

- (ir) $(x \prec x) \forall x \in X$; irreflexive
- (as) $(x \prec y) \Rightarrow (y \not\prec x)$; asymmetric
- (tr) $(x \prec y) \land (y \prec z) \Rightarrow (x \prec z)$; transitive
- (co) $(x \prec y) \lor (y \prec x) \forall x \neq y$. complete

Habitually we will avoid using the symbol $\prec$ if the binary relation will be reflexive or symmetric. In this case we will rather use the more generic symbol $R$.

- (rc) $(xRx) \forall x \in X$; reflexive
- (sy) $(xRy) \Rightarrow (yRx)$; symmetric
- (an) $(xRy) \land (yRx) \Rightarrow x = y$. antisymmetric
1.2. Let \((X, \prec)\) be a set with a binary relation \(\prec\). We call \(\prec\) a preference if axiom (as) ( asymmetric) holds. We call \(\prec\) a partial (strict) order if axioms (iv) and (tr) or equivalently (as) and (tr) hold. \((X, \prec)\) is called a poset. We call \(\prec\) a linear (strict) order if \(\prec\) is a partial (strict) order which is complete.

Let \((X, \prec)\) be a poset. A (bad) habit is to say that two elements \(x\) and \(y\) are indifferent if \(x \neq y \land (y \neq x)\). We will denote this relation with \(x \sim y\). We prefer to call them incomparable. We will define indifferent two elements \(x\) and \(y\) if \(z \prec x \iff z \prec y\) and \(x \prec z \iff y \prec z\) \(\forall z \in X\), and denote it with \(x \approx y\). \(\approx\) is obviously an equivalence, while \(\sim\) is only reflexive and symmetric, but not necessarily transitive.

It is immediate that \(x \approx y \Rightarrow x \sim y\), but not conversely.

It can be easily verified that:

\[\sim \text{ transitive } \iff \prec \text{ negatively transitive } \iff I \approx I.\]

1.3. Let \(\prec\) be a partial order. We can call left-set or lower-set (right-set or upper set) of origin \(x\) the set \(\{y : y \prec x\} = x^<_\prec (\{y : x \prec y\} = x^>_\prec)\).

2. Topology and semicontinuous utility functions on \((X, \prec)\)

2.1. We consider now a set provided with a topology \(\tau\) and a partial order \(\prec\): \((X, \prec, \tau)\).

We will call utility function on \((X, \prec)\) a real function \(u(x)\) such that \(x \prec y \Rightarrow u(x) < u(y)\). Obviously, if there exists a utility function \(u\) on \((X, \prec)\), then \(x \sim y \Rightarrow u(x) = u(y)\). Therefore \(I\) is transitive, which means \(I \approx I\) and \(\prec\) negatively transitive.

There exists a utility function on a poset \((X, \prec)\) only if \((X/ \approx, \prec)\) is linearly ordered.

2.2. Recall that \(f : (X, \tau) \to R\) is called upper (lower) semicontinuous if \(\{x : f(x) < (>)y\}\) is open for each \(y \in R\). It is called continuous if both semicontinuities hold.

If there exists an upper semicontinuous utility function on \((X, \prec, \tau)\), \(\prec\) must be upper semicontinuous. It means that the topology \(\tau\) has to be finer than the left set topology (generated by \(x^<_\prec\)). So, for the existence of an upper semicontinuous utility function on \((X, \prec, \tau)\), it is necessary to suppose that \(x^<_\prec\) is open for every \(x\).

If we add, to the necessary conditions (for \(\prec\)) of upper semicontinuity and linear order on \(X/\approx\), the hypothesis that \(\tau\) is 2-nd countable, we will be able to build an upper semicontinuous utility function.

2.3. As an example of a “good use” of the axioms, we will go in details considering a famous theorem due to Trout Rader [7], whose proof has been recently discussed and disapproved by G. B. Mehta [6].