A STUDY OF HEAT CONDUCTION IN STRUCTURAL CERAMIC MATERIALS. PART III. MATHEMATICAL MODEL OF MEASURING CELL

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The process of heat propagation in a specimen is considered in an approximation of a one-dimensional heat flow with side leakages of heat. They are modelled as a function of the heat sources (sinks). In the given case the heat will be negative because it will be lost.

Let a longitudinal heat flow passing through a specimen be directed along the axis $x$ and a transverse flow be directed over the surface of the specimen along the axis $y$ (Fig. 8). Then the temperature field in the specimen under stationary heating will be describable, by the Fourier law, by a nonlinear one-dimensional differential equation

$$
\frac{d}{dx} \left[ \lambda(T) \frac{dT(x)}{dx} \right] = f(x, r_0),
$$

where $f(x, r_0)$ is the source function and $r_0$ is the distance from the center of the specimen to its side surface.

In order to solve this equation we should know the boundary conditions and the source function, which in their turn can be found from experimental data. It is known from [132] that the boundary conditions can be of at least three types, though their variety is not limited to them. In our case we will use the following boundary conditions:

- $T(x_i) = T_i,$
- $\lambda \frac{dT}{dx} \bigg|_{x=r_\ell} = q_\ell,$

where $q_\ell$ is the density of the heat flow in the end calorimeter.

It can be seen from Fig. 8 that $T_i$ is measured by the first thermocouple on the surface of the specimen; $q_\ell$ is determined with the help of a flow-type microcalorimeter.

The source function is first computed at every point $x_i$, i.e.,

$$
f(x_i, r_0) = pq_v(x_i),
$$

where $p$ is the proportion of the perimeter of the specimen to the cross-sectional area, $q_v(x_i)$ is the density of the radial heat flow at an individual point $x_i$ measured by the side calorimeter, and $r_\ell$, $r_\ell'$ are the radial coordinates of the positions of the thermocouples.

The found discrete values of $f(x_i, r_0)$ are approximated by curves from which we obtain $f(x, r_0)$.
The nonlinear one-dimensional differential equation is used in an implicit identification method. For this purpose we solve it numerically at each iteration step in the procedure of searching for the minimum of the quality criterion. In this case the difference analog is as follows:

\[ T_{i+1} - T_i = \frac{1}{2} \left( \frac{T_{i+1} - T_i}{X_{i+1} - X_i} - f(x_i, r_0) \right) = 0. \]

Let us consider the solution procedure. Suppose that we have a running estimate of the thermal conductivity \( \hat{\lambda}(T) \). Then we should calculate \( T \) for the specified boundary conditions and the source function \( f(x, r_0) \). The solution can be found by the mesh method [133]. Since the differential equation is linear, we use iterations called ‘the iterations of the solution of the equation’ to distinguish them from the iterations of minimization of the quality criterion. The solution mesh in an interval \([x_1, x_N]\) is specified in a uniform form with a step \( h \) and points \( \{x_i\}_N \), where \( N < 100 \). For the first iteration we choose a temperature profile \( T^0(x) = \text{const} \); for the specified \( \hat{\lambda}(T) \) we use it to plot the function \( \hat{\lambda}^0(x) \). As a result, we obtain a system of linear algebraic equations with respect to the vector \( \{T_i\}_N \). Its solution gives us the values of \( \{x_i\}_N \) and a temperature profile \( T^1(x) \). It is used to plot \( \hat{\lambda}^1(x) \), and the procedure is repeated.

Practical calculations have shown that ten iterations of the differential equation are required for providing a sufficient accuracy.

The identification procedure is performed by explicit and implicit methods of solving IPHC. The former is used to determine the criteria for regulation of the descriptive features and the initial approximations of the value of the thermal conductivity. The second method is predominantly used for processing the packet of experimental data in the evaluation of the effective heat conduction.

In the explicit method of identification the thermal conductivity \( \hat{\lambda}_i(T) \) is obtained from the estimate \( \hat{\lambda}_i(x) \) computed from the proportion

\[ \hat{\lambda}_i(x) = \frac{\hat{q}(x)}{d\hat{T}/dx}, \]

where \( \hat{q}(x) \) is the estimate of the density of the longitudinal heat flow and \( d\hat{T}/dx \) is the estimate of the temperature gradient as a function of \( x \in (x_1, x_n) \).

Consequently, in order to obtain \( \hat{\lambda}_i(x) \) we should determine \( \hat{q}(x) \) and \( d\hat{T}/dx \). Since the function \( T = T(x) \) is a one-to-one dependence, we can write the following expression:

\[ \hat{\lambda}_i(T) = \hat{\lambda}_i[T^{-1}(x)]. \]

Thus, we obtain a unique value of \( T \) for every \( x \), and a dependence \( \hat{\lambda}_i(T) \).

The estimate \( \hat{q}(x) \) of the density of the longitudinal heat flow \( q(x) \) is determined at each point \( \{x_i\}_N \) from the axial and radial heat flows. First we divide the sum of the heat flows by the area of the cross section of the specimen and get \( \hat{q}(x), i = 1, n \). Then we approximate \( \hat{q}(x_i) \) by the least squares method [133] and find a continuous estimate of \( \hat{q}(x) \). For its approximation we use functions of the following classes:

\[ ae^{bx^2} + be^{bx} + c = q(x), \]

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\[ ax^2 + bx + c = q(x), \]

The estimate of \( d\hat{T}/dx \) is obtained by approximating the values \( T(x_i), i = 1, n \), by the least squares method and by analytical differentiation of the obtained dependence. The approximation is based on functions of the same classes the number of parameters of which is specified to be equal to or less than the number of the measured points. Having found the estimates for \( \hat{q}(x) \) and \( d\hat{T}/dx \), we determine \( \hat{\lambda}_i(x) \) at the requisite points \( x_i \) and \( \hat{\lambda}_i(T) \).

Since there exists a wide choice of the number of approximating functions for \( q(x) \) and \( dT/dx \) and of their classes, we should develop a criterion of the quality of the estimation of \( \hat{\lambda}_i(T) \). A unique criterion can be the measure of the deviation of the measured temperatures \( \{T_i\}_N \) at the points \( \{x_i\}_N \) from the temperatures computed in the solution.