THE PROBLEM OF THE SLOW ROTATION OF A SPHERE IN AN UNBOUNDED GAS

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The problem of the slow rotation of a sphere in an unbounded gas is solved at arbitrary Knudsen numbers. The kinetic equation with collision integral in the form of the Bhatnagar-Gross-Krook model (BGK-model) describing the state of the surrounding medium is solved by the Lees method, all the moments of the distribution function which ensure the asymptotic passage of the distribution function into the Chapman-Enskog distribution function at large distances from the sphere being taken into account. In the particular case of moderately large spherical particles a value of the isothermal slip coefficient similar to the exact value is obtained.

The problem of the behavior of a gas at arbitrary Knudsen numbers remains one of the important problems of the kinetic theory of a rarefied gas [1—3]. This is associated both with the great practical importance of these problems and with serious difficulties arising in connection with their theoretical analysis. So far, the Lees method [4—9] has been the basic method used for the analytic solution of such problems. This method has proven its practical importance. However, the usual variant involves a serious theoretical difficulty associated with the behavior of the distribution function at large distances. This difficulty relates to the fact that the number of moments usually chosen is insufficient to trace the transition of the distribution function to the Chapman-Enskog distribution. At the same time, increasing the number of moments leads to considerable mathematical difficulties in solving problems of this type. In particular, in [8] in considering the problem of the rotation of a sphere of radius R at arbitrary Knudsen numbers only one moment was retained. (We recall that taking the influence cones into account leads to the doubling of the effective number of moments so that in [8] two moments were actually taken into account, see below.)

In the present paper we will consider the problem of the rotation of a sphere at arbitrary Knudsen numbers with allowance for twice as many moments $(c_\phi$ and $c_\phi^2$), which makes it possible to trace the transition of a given distribution function to the Chapman-Enskog distribution. We note that even in the case of a fine particle (at large Knudsen numbers) the distribution function must go over to the Chapman-Enskog function at large distances from the particle since far away from the particle the local Knudsen number will be small [1].

1. FORMULATION OF THE PROBLEM

We will consider the case of a slowly rotating sphere, i.e., we will assume that its linear rotation velocity $\omega R$ is much lower than the thermal velocity of the gas molecules $(\omega R << (2kT/m)^{1/2})$. In this case the distribution function can be linearized, i.e., represented in the form:

$$F(r, v) = f_0(1 + \Phi(r, v))$$

$$f_0 = m(n/2\pi kT)^{3/2} \exp(-c^2), \quad c = \sqrt{m/2kT} v$$

Here, $m$ and $v$ are the mass and the velocity of the gas molecules, $n$ is the number density of the molecules, $T$ is the gas temperature, and $r = |r|/R$.

In order to find the function $\Phi(r, v)$ we will use a kinetic equation with collision integral in the form of the BGK-model [1, 2]. In the linearized variant and with allowance for the symmetry of the problem this equation is

$$c\nabla \Phi = e (2c_\phi G_\phi - \Phi)$$  \hspace{1cm} (1.1)

In the notation of [1], in Hilbert space the scalar product is determined by the formula

$$\langle \Psi, \Phi \rangle = \int \exp(-c^2) \Psi \Phi \frac{d^2c}{\pi^{3/2}}$$

and then, in this notation, the projection of the dimensionless macroscopic gas velocity takes the form $G_\phi = (c_\phi, \Phi)$. 

The parameter $\epsilon$ can be chosen using the condition that the Chapman-Enskog distribution function $f_{ch} = f_0(1 + 2c_\epsilon G_\epsilon - b_1 c_\epsilon \Pi_{\phi \phi})$ must satisfy Eq. (1.1). The component $\Pi_{\phi \phi}$ of the stress rate tensor is

$$\Pi_{\phi \phi} = \left[ r \frac{\partial}{\partial r} \left( \frac{G_\phi}{r} \right) + \frac{1}{r \sin \theta} \frac{\partial G_\phi}{\partial \phi} \right]$$

and the coefficient $b_1$, for example for rigid sphere molecules, can be expressed in terms of the average free path $\lambda$ and is equal [2] to $5\sqrt{\pi} \lambda/4$. Thus, $\epsilon = 2R/b_1 = 8/(5\sqrt{\pi} \text{Kn})$, where Kn $= \lambda/R$ is the Knudsen number.

Equation (1.1) is solved by the Lees method [4, 5] in the form of a function discontinuous in velocity space

$$\Phi(r, \theta) = \eta(\pm)\Phi^+ + \eta(\mp)\Phi^-$$

$$\eta(\pm) = \frac{1}{2} (1 \pm \text{sign}(c_r - c_\rho))$$

$$\chi_p = \arcsin (r^{-1})$$

$$\gamma = \cos \chi_p = \sqrt{1 - r^{-2}}$$

where $\text{sign}(c_r - c_\rho)$ is the sign function.

The relation between the Cartesian and spherical coordinates has the form $c_r = c \cos \chi, c_\rho = c \sin \chi \cos \varphi$, and $c_\phi = c \sin \chi \sin \varphi$.

Inside and outside the influence cone the angle $\chi$ must satisfy the relations $0 \leq \chi \leq \chi_p$ and $\chi_p \leq \chi \leq \pi$.

We note that in study [8] the $\Phi^+$ were chosen so that

$$\Phi^+ = A^+ (r, \theta) c_\phi$$

This distribution function does not ensure the correct Chapman velocity distribution of the gas molecules at distances greater than $\lambda$. Therefore, in order to remove this constraint we choose $\Phi^+$ with a larger number of moments, namely, in the form:

$$\Phi^+ = c_\phi A_1(r, \theta) + c_r c_\rho B_1(r, \theta), \quad \Phi^- = c_\phi A_2(r, \theta) + c_r c_\rho B_2(r, \theta)$$

As $r \to \infty$ the functions $A_2(r, \theta)$ and $B_2(r, \theta)$ must decrease and on the surface of the sphere (at $r=1$) $f=f_0(1 + c_\phi A_1(1, \theta) + c_r c_\rho B_1(1, \theta)$.

If we assume that the molecules are diffusely reflected with a Maxwellian distribution $f_M = f_0(1 + 2c_\phi \omega_0 \sin \theta)$, then the boundary conditions for the functions $A_1(r, \theta)$ and $B_1(r, \theta)$ will be

$$A_1(1, \theta) = 2c_\phi \omega_0 \sin \theta, \quad B_1(1, \theta) = 0$$

where $\omega_0 = \omega R(m/(2kT))^{1/2}$.

When the symmetry of the problem and the boundary conditions (1.3) are taken into account, clearly, the solutions can be sought in the form:

$$A_i(r, \theta) = a_i(r) \sin \theta, \quad B_i(r, \theta) = b_i(r) \sin \theta, \quad i = 1, 2$$

2. THE MOMENT EQUATIONS AND THEIR SOLUTIONS

The moment equations for the functions $a_i(r)$ and $b_i(r)$ can be obtained after multiplying Eq. (1.1) successively by $c_\phi, c_r c_\rho, c_\rho S_{3/2}^{(1)}(c^2)$, and $c_\phi c_r S_{5/2}^{(1)}(c^2)$ and then integrating over the entire velocity space (here, $S_{3/2}^{(1)}(c^2)$ and $S_{5/2}^{(1)}(c^2)$ are Sonine polynomials). We note that these moments are mutually orthogonal. As a result, the system of moment equations takes the form:

$$\frac{d}{dr} \left( r^4 \tau_{rr}^i \right) = 0$$

$$\frac{1}{r^3} \frac{d}{dr} \left[ r^3 \left( \frac{\sqrt{\pi}}{2} (a_1 + a_2) + \frac{\sqrt{\pi}}{4} (5\gamma^3 - 3\gamma^5)(a_1 + a_2) + (1 - 3\gamma^4 + 2\gamma^6)(b_1 - b_2) \right) \right] =$$

$$\frac{1}{r} \left( 2\sqrt{\pi} (a_1 + a_2) - \frac{\sqrt{\pi}}{4} (15\gamma - 10\gamma^3 + 3\gamma^5)(a_1 + a_2) + \frac{2}{r_0^2} (b_1 - b_2) \right) = -4\sqrt{\pi} \epsilon \tau_{rr}$$