HYDRAULIC JUMP STRUCTURES IN THE PRESENCE OF AN ICE SHEET

I. B. Bakholdin

Jumps of the bore type arising in a fluid layer with an ice sheet are investigated. These jump structures are considered for a determining mechanism in the form of dispersion due to the presence of an ice sheet. For this purpose a generalized Korteweg-de Vries equation [1] is used. The structure of these jumps consists of a wave zone that expand with time. On the boundary of the wave zone there are transitions between uniform and periodic states which can be locally considered as jumps. Among them are jumps which can be regarded as steady in the coordinate system moving with the boundary of the wave zone. These are jumps between a sequence of solitons and a uniform state (jumps of soliton type) on the boundary of the wave zone and jumps between periodic and uniform states (jumps with radiation). In addition, there are jumps which are unsteady even from the standpoint of a local analysis. In order to investigate the effect of dissipation processes on the jumps considered a system of generalized Boussinesq equations is derived with allowance for bottom slope and bottom and ice friction. The jump damping process is investigated numerically. This system of equations also makes it possible to investigate undamped jumps of the flood-water wave type.

All possible types of jumps were analyzed in [2] for the general case of symmetric dissipationless models with dispersion and in [3] for models of the generalized Korteweg-de Vries equation type. In this case by jumps is meant all possible transitions between uniform, periodic, quasiperiodic, and stochastic states. The aim of the present study is to express these results with reference to long waves on the surface of the fluid with an ice cover and analyze the effect of weak dissipative processes occurring in nature. Jumps can develop when a tidal wave arrives at the shelf (tidal bore) or at the beginning of discharge from a dam. The investigation for a dam is particularly important in connection with the fact that the severe bending can result in the fracturing and subsequent break up of the ice. This can have an adverse effect on the environment of the region in winter. This fracturing can be obviated by properly controlling the discharge process. On the other hand, in spring a flood water wave of considerable intensity may be specially created to rid the basin of ice.

1. INVESTIGATION OF JUMPS WITHOUT ALLOWANCE FOR DISSIPATION

In order to describe small-amplitude long waves on a fluid surface beneath ice [1] we will use the generalized Korteweg-de Vries equation in the dimensional form:

\[ \eta_t + \frac{3}{2} \sqrt{\frac{g H}{g}} \eta \eta_x + \sqrt{g H} \eta_x + \frac{1}{2} \sqrt{g H} \left( \frac{H^2}{3} + H \eta - \frac{\sigma_{0}}{g} \right) \eta_{xx} + \frac{1}{2} \sqrt{g H} \left( \frac{H}{g} \right) \eta_{xxx} = 0 \] (1.1)

Here, \( x \) is the horizontal axis, \( g \) is the acceleration of gravity, \( \eta \) is the elevation of the fluid surface with respect to zero level, \( H \) is the undisturbed fluid depth, \( h \) is the thickness of the ice sheet, \( v \) is Poisson’s ratio, \( E \) is Young’s modulus, \( \rho \) is the density, and \( \sigma_{0} \) is the initial horizontal stress in the ice.

As a result of discharge, in the neighborhood of the dam the water level will differ from that before discharge began. The Riemann problem arises. In order to apply the results of study [3] to the problem of a bore developed as a result of initial discharge from a dam it is necessary to know the value of the increase in the level after discharge. If the velocity of the discharged water or the water flow rate across the dam are known, then the surface elevation can be determined on the basis of the Riemann invariant conservation condition [4]

\[ u + \sqrt{2g(H + \eta)} = \sqrt{2gH} \eta \]

where \( u \) is the depth-average fluid velocity.

The quantity \( \eta_{0} \) is of importance since it is related with the ice stresses. The maximum stresses are reached on the upper or lower boundary of the ice and can be determined from the formula

\[ \sigma_{max} = \frac{E h^3}{12(1 - v^2)\rho} \eta_{xxx} \]
Here, $R$ is the radius of curvature of the middle surface of the ice plate. In practice, we can assume that this is the radius of curvature of the fluid surface. When $|\sigma_{xx}| \approx 10^8$ Nm$^{-2}$ the ice breaks up.

We will consider the above problems as problems of the evolution of an initial discontinuity, i.e., of the evolution of initial data of the smoothed step type. With time the solution becomes self-similar and depends only on the initial discontinuity amplitude; therefore, the nature of the smoothing is of no importance. In fact, the function $[1 + \tanh (-v/l)]$ was used.

To obtain numerical solutions we used a three-layer central-difference scheme. For the purpose of reducing the number of the problem parameters varied and using the computation method more effectively, we will go over to the moving coordinate system by substituting $x \rightarrow x - \frac{(gH)}{t}$ and then go over to the dimensionless coordinates $t = T/t'$, $x = Xx'$, and $\eta = A\eta'$ so that

$$\frac{3 \sqrt{gH \Delta T}}{2} = 1, \quad \frac{\sqrt{gH}}{6} + \frac{hH}{2} - \frac{\sigma_{hh}}{2g\rho} \cdot \frac{T}{X^3} = 1, \quad \frac{1}{2} \left( \frac{H}{g} \right) \frac{Eh^3}{12(1 - \nu^2) \rho} \frac{T}{X^3} = 1$$

The equation takes the normalized form:

$$\eta' + \eta' \eta + \eta'_{x} + \eta'_{x,x} = 0 \quad (1.2)$$

From (1.1) we can see that the sign of $\eta'_{x,x}$ in (1.2) depends on the initial stresses, the fluid depth, and the ice sheet thickness. The amplitude of the initial discontinuity is a single variable parameter. In order for the normalized form of Eq. (1.2) to coincide completely with that used in [3] when the sign of $\eta'_{x,x}$ is minus it is necessary to replace $\eta$ by $-\eta$ and $x$ by $-x$.

In Fig. 1 we have reproduced examples of the graphs of $\eta'(x')$ for all possible types of solutions. These are solutions with soliton-like jumps, jumps with radiation, and jumps with a stationary structure.

In the first case (Fig. 1a) intervals with constant values of $\eta$ are separated by a wave zone. With time the wave envelope becomes self-similar (function of $x/t$). On one of the boundaries of the wave zone with time the wave train tends to a sequence of solitons.

In the second case (Fig. 1b) the wave zone also expands self-similarly with time, but here the wave zone consists of an interval with a centered simple envelope wave and an interval with constant wave amplitude. On the boundary of the wave zone there is a local jump between the interval with a constant value of $\eta$ and that with a constant envelope wave amplitude. The structure of this jump is constant.

In the third case (Fig. 1c) the wave zone can be divided into two parts between which there is a local jump with nonstationary structure, on either side of this jump the wave zone intervals are chaotic. With decrease in the amplitude of the initial discontinuity the nonstationary properties of the jump grow weaker, to the right of the jump the wave amplitude decreases, and the solution becomes similar to that with a soliton-like jump.

The type of jump depends on the amplitude of the initial discontinuity [3]. Let $\eta' \rightarrow 0$ as $x' \rightarrow 0$ and $\eta' \rightarrow \eta_1$ as $x' \rightarrow -\infty$. To determine the type of jump it is necessary to consider the solution of the system of equations describing the dispersion curve and the straight line corresponding to the phase velocity $U$ of the jump

$$\omega = \eta_1 k + k^3 + k^5, \quad \omega = Uk \quad (1.3)$$

The value of $U$ can be determined from a numerical experiment. In the case of jumps with stationary structure the average velocity is understood.

In the case of a minus sign in Eq. (1.2) there exists a certain critical value of $\eta_{1,2} \approx 0.417$ determined from an analysis of the solutions of (1.3). When $\eta_1 > \eta_{1,2}$ the quantity $k$ is complex and a jump with radiation develops. For $\eta_1 < \eta_{1,2}$ the quantity $k$ is purely imaginary and the jump is soliton-like.

In the case of a plus sign two types of solution are also possible. For a high amplitude a jump with radiation develops and for a fairly low amplitude a jump with a structure of the nonstationary type. A theoretical investigation reveals an interval $\eta_{2,5} < \eta_2 < \eta_{2,2}$, on which both types of solution can exist. The value of $\eta_{2,5} \approx 0.522$ corresponds to the point of transition from the real to complex values of $k$. When $\eta_2 > \eta_{2,5}$ a jump with radiation is possible [2]. When $\eta_{2,5} \approx 0.887$ the jump velocity coincides with the minimum group velocity. For $\eta_2 > \eta_{2,5}$ in the region to the left of the jump waves cannot be radiated and therefore a jump with nonstationary structure cannot exist. By means of a numerical analysis it was found that $\eta_{2,2} = 0.68$ for the point of transition from jumps with nonstationary structure to steady jumps with radiation.