A SHARP SLENDER CONE IN AN INCOMPRESSIBLE FLOW

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The range of applicability of the asymptotic solution of the problem of incompressible flow past a slender cone is studied.

The asymptotic problem of the flow in the vicinity of a sharp conical nose can be reduced to the solution of the Legendre equation with noninteger eigenvalues $1 < n < 2$ which can be represented in hypergeometric series form. The calculation of the series, like the straightforward, computer-aided solution of the boundary value problem, does not present a particular problem; however, in the case of a slender cone with a small semivertex angle $\theta_0$, it is somewhat complicated owing to the singularity of the solution as $\theta_0 \to 0$. An asymptotic solution for this case was derived in [1]; the eigenvalue spectrum for this problem was also found in [2]. In what follows, we analyze the range of applicability of the asymptotic solution and demonstrate that it can be used for cones with semivertex angles $\theta_0 \leq 20^\circ$.

In a spherical reference frame $r, \theta, \phi$ centered on the nose of the body and having the ray $\theta=0$ opposed to the freestream velocity, as shown in Fig. 1, the equation for the velocity potential has the form:

$$ r \frac{\partial}{\partial r} r^2 \frac{\partial \Phi}{\partial r} + \frac{r}{\sin \theta} \frac{\partial}{\partial \theta} \left[ \sin \theta \frac{\partial \Phi}{\partial \theta} \right] = 0 $$

$$ V_r = \frac{\partial \Phi}{\partial r}, \quad V_\theta = \frac{1}{r} \frac{\partial \Phi}{\partial \theta}, \quad V_\phi = \frac{1}{r \sin \theta} \frac{\partial \Phi}{\partial \phi} = 0 $$

The following family of solutions corresponds to the problem formulated

$$ \Phi = C U_r r^n \Theta(\theta), \quad C = \text{const}; \quad V_r = n C U_r r^{n-1} \Theta'(\theta), \quad V_\phi = C U_r r^{n-1} \Theta'(\theta) $$

The constant $C$ has the dimensionality $L^{1-\alpha}$, where $L$ is a scale length; it should be determined by matching this local solution with the global one. The $\Theta(\theta)$ dependence is obtained from the solution of the following eigenvalue problem

$$ (\Theta' \sin \theta)' + n(n + 1) \sin \theta \Theta = 0; \quad \Theta_\theta - \Theta = 0, \quad \theta = 0, \pi - \theta_0 $$

The substitution $t = \cos \theta$ in Eq. (3) leads to the canonical Legendre equation

$$ \frac{d}{dt} \left[ (1 - t^2) \frac{d\Theta}{dt} \right] + n(n + 1) \Theta = 0 $$

$$ \theta = 0, \quad t = 1, \quad \frac{d\Theta}{dt} = -\sin \theta \frac{d\Theta}{dt} = 0; \quad \theta = \pi - \theta_0, \quad t = 0, \quad \frac{d\Theta}{dt} = 0 $$

For integer $n$, the solution of this equation is given by the Legendre polynomials $P_n(t), P_n(1)=1$. Two of these polynomials

$$ \Theta_1 = -P_1 = -t, \quad n = 1, \quad \theta_0 = 0; \quad \Theta_2 = -P_2 = \frac{3}{2} t^2 + \frac{1}{2}, \quad n = 2, \quad \theta_0 = \frac{\pi}{2} $$

correspond to the undisturbed stream and the flow in the vicinity of the stagnation point on a flat-faced body, respectively. The interval $1 < n < 2$ corresponds to the cone semivertex angles $0 < \theta_0 < \pi/2$. At $n > 2$, the eigenfunctions incorporate additional rays (apart from the axis of symmetry and the conical surface) on which the condition $\partial \Theta / \partial \theta = 0$ is fulfilled. Thus, for $\Theta = P_1$ these rays are $\Theta_1 = 75^\circ$ and $\Theta_2 = 135^\circ$. These solutions describe the flow within the angles formed by these rays and may be of interest only as the terms of the series-expansion of the solution for a finite-length cone.
For noninteger \( n \) and \(|t| < 1\), Eq. (4) has particular solutions which are regular at one of the singular points \( t = \pm 1 \) and singular at the other point. The formal solution of the problem formulated is given by a Legendre function of the first kind or a hypergeometric series:

\[
\Theta(t) = -P_n(t) = -F(-n, n + 1, 1, 1 - t/2), \quad P_n(1) = 1
\]  

(6)

A set of Legendre functions\(^1\) with the opposite sign is also the solution of the problem (4), (5) in the region \( 0 < \theta < \pi - \theta_0 \); the eigenvalue \( n \) is then determined from condition (5).

For slender cones \((\theta_0 \ll 1, n = 1)\), the direct solution of the problem is somewhat complicated by the fact that the conical surface is located within the domain of influence of the singular point \( t = 1 \). However, in this case the asymptotic solution of Eq. (4) can be obtained by linearizing this equation. The equation is linear in \( n \) and nonlinear with respect to the eigenvalue problem since both \( \theta \) and \( n \) depend on \( \theta_0 \).

The following solution was derived in [1] for cones with small semivertex angles:

\[
\Theta = -t + \varepsilon \Theta_1 + \ldots, \quad n = 1 + \varepsilon, \quad \varepsilon < 1
\]  

(7)

Substituting (7) in (5), retaining terms of the order of \( \varepsilon \) only, eliminating the singularity as \( t \to 1 \) by an appropriate choice of constants in the solution, and satisfying the condition \( \Theta_1(1) = 0 \), we easily obtain

\[
\Theta_1 = t [\ln 2 - \ln (1 + t)] + 1 - t
\]  

(8)

Since at small \( \theta_0 \) we have \( 1 + \tau_0 = \theta_0^2 / 2 \), from the condition \( \partial \Theta / \partial \varepsilon = 0 \) there follows

\(^1\)These functions were tabulated in [3] but with a fairly large step in \( n \) and on the interval \( 0 \leq \theta < \pi/2 \) only.