TOPOLOGICAL QUANTUM FIELD THEORIES

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To René Thom on his 65th birthday.

1. Introduction

In recent years there has been a remarkable renaissance in the relation between Geometry and Physics. This relation involves the most advanced and sophisticated ideas on each side and appears to be extremely deep. The traditional links between the two subjects, as embodied for example in Einstein's Theory of General Relativity or in Maxwell's Equations for Electro-Magnetism are concerned essentially with classical fields of force, governed by differential equations, and their geometrical interpretation. The new feature of present developments is that links are being established between quantum physics and topology. It is no longer the purely local aspects that are involved but their global counterparts. In a very general sense this should not be too surprising. Both quantum theory and topology are characterized by discrete phenomena emerging from a continuous background. However, the realization that this vague philosophical view-point could be translated into reasonably precise and significant mathematical statements is mainly due to the efforts of Edward Witten who, in a variety of directions, has shown the insight that can be derived by examining the topological aspects of quantum field theories.

The best starting point is undoubtedly Witten's paper [11] where he explained the geometric meaning of super-symmetry. It is well-known that the quantum Hamiltonian corresponding to a classical particle moving on a Riemannian manifold is just the Laplace-Beltrami operator. Witten pointed out that, for super-symmetric quantum mechanics, the Hamiltonian is just the Hodge-Laplacian. In this super-symmetric theory differential forms are bosons or fermions depending on the parity of their degrees. Witten went on to introduce a modified Hodge-Laplacian, depending on a real-valued function \( f \). He was then able to derive the Morse theory (relating critical points of \( f \) to the Betti numbers of the manifold) by using the standard limiting procedures relating the quantum and classical theories.
With this model of super-symmetric quantum mechanics rigorously understood, Witten then went on to outline the corresponding ideas for super-symmetric quantum field theories. Essentially such quantum field theories should be viewed as the differential geometry of certain infinite-dimensional manifolds, including the associated analysis (e.g. Hodge theory) and topology (e.g. Betti numbers).

Great caution has of course to be used in such infinite-dimensional situations but, taking one's cue from physics on the one hand and topology on the other hand, it is possible to make intelligent guesses and conjectures. There is now ample evidence in favour of many of these conjectures, a number of which have been rigorously established by alternative methods. This applies for example to results concerning "elliptic cohomology" [17] and to the topic I shall discuss in detail in this paper.

Perhaps a few further comments should be made to reassure the sceptical reader. The quantum field theories of interest are inherently non-linear, but the non-linearities have a natural origin, e.g. coming from non-abelian Lie groups. Moreover there is usually some scaling or coupling parameter in the theory which in the limit relates to the classical theory. Fundamental topological aspects of such a quantum field theory should be independent of the parameters and it is therefore reasonable to expect them to be computable (in some sense) by examining the classical limit. This means that such topological information is essentially robust and should be independent of the fine analytical details (and difficulties) of the full quantum theory. That is why it is not unreasonable to expect to understand these topological aspects before the quantum field theories have been shown to exist as rigorous mathematical structures. In fact, it may well be that such topological understanding is a necessary prerequisite to building the analytical apparatus of the quantum theory.

My comments so far have been of a conventional kind, indicating that there may be interesting topological aspects of quantum field theories and that these should be important for the relevant physics. However, we can reverse the procedure and use these quantum field theories as a conceptual tool to suggest new mathematical results. It is remarkable that this reverse process appears to be extremely successful and has led to spectacular progress in our understanding of geometry in low dimensions. It is probably no accident that the usual quantum field theories can only be renormalized in (space-time) dimensions \( \leq 4 \), and this is precisely the range in which difficult phenomena arise leading to deep and beautiful theories (e.g. the works of Thurston in 3 dimensions and Donaldson in 4 dimensions).

It now seems clear that the way to investigate the subtleties of low-dimensional manifolds is to associate to them suitable infinite-dimensional manifolds (e.g. spaces of connections) and to study these by standard linear methods (homology, etc.). In other words we use quantum field theory as a refined tool to study low-dimensional manifolds.

Now quantum field theories have, because of the difficulties involved in constructing them, often been described axiomatically. This identifies their essential structural