MATHEMATICAL SIMULATION OF GRAVITATIONAL WAVES IN THE OCEAN IN THE APPROXIMATION OF "SHALLOW WATER"

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Problems of the behavior of gravitational waves in the approximation of "shallow water" — motion of a solitary wave over the water surface, exit of a wave to the shore, passage of a solitary wave over a submerged rock — are solved. The solution of the first problem showed that in modeling the motion of a solitary (soliton-type) wave the "shallow water" approximation breaks down at a ratio of water depth to wavelength equal to 0.3. An analysis of the results of solution of the second problem indicates that the "shallow water" approximation cannot be used for calculation of the height of a wave in its exit to the shore but it can be used for estimation of the distance from the shore where the wave is turned over. It follows from the solution of the third problem that the "shallow water" approximation is suited only for obtaining a qualitative picture of the distortion of the profile of a wave in its motion over a rock.

Several stages can be distinguished in a theoretical study of the tsunami phenomenon [1]:
1) description of the source of the disturbance and the motion of the water under the action of external forces (determination of the initial profile of the wave);
2) formation of a wave structure, which then moves as a whole;
3) propagation of the wave in a deep-water portion of the ocean;
4) incidence of the wave on a shallow-water near-shore portion of the ocean;
5) reflection of the wave from an uneven part of the bottom and artificial submerged structures.

In this paper attention was concentrated on items 2-5. It was assumed that the external action is caused by the fall of a celestial body into the water rather than by motion of the bottom (as is usually adopted) [2].

A tsunami can apparently arise as a result of the fall of asteroids and comets into water basins. Numerical simulation of the fall of an asteroid with a radius of 10 km, a substance density of 2.5 g/cm³, and an energy of $6 \times 10^7$ Mton in TNT equivalent into an ocean with a depth of 5 km showed that about 12% of the kinetic energy of the body is transferred to a water layer, thus leading to the appearance of a powerful wave with a height of 25–35 km that is capable of disastrous consequences over the entire territory of the earth. In fact, the fall of such an asteroid to earth is unlikely. However, estimates of the consequences in the fall of an asteroid with a radius of 0.2 km showed that this event can be accompanied by catastrophic phenomena, because here a wave with an initial amplitude of about 0.3–0.8 km is formed [3].

The length of tsunami waves in the region of generation is usually close to the size of the site of the disturbance. Here it can amount to only several kilometers. The initial disturbances are grouped into a system consisting of two or three waves following each other in the open ocean. A difficulty in modeling this stage (item 2) is associated with the impossibility of using the "shallow water" approximation.

As the waves move, they become not too pronounced: their height (i.e., the vertical distance from the crest to the trough) amounts to several meters, and the length can reach tens or even hundreds of kilometers [4]. Even the deepest parts of the ocean turn out to be shallow for them, and modeling of this stage does not present special difficulties because the "shallow water" approximation can be used. The velocity of the wave is determined by the Lagrange formula: $v = \sqrt{gh}$. With a mean depth of the Pacific Ocean of about 4000 m, the theoretically calculated...
velocity of a tsunami wave is 716 km/h. This seems unlikely, but the maximum measured velocity of a tsunami wave was even higher – about 1000 km/h. In fact, the velocity of the majority of tsunami waves is somewhat below the theoretical value and ranges from 400 to 500 km/h.

On approaching the shore, a tsunami can grow from 1–2 m in the open ocean to several tens of meters on the shore depending on the coastal relief of the bottom and the shape of the shore line. But, the chief factor with which an increase in the height of the wave is associated is the decrease in the depth of the ocean. The latter can be calculated by the Airy–Green formula \( h_{sh} = h_m^{\sqrt{H_m/H_{sh}}} \) [4].

Having reached a shallow-water shelf, the wave becomes higher, heaves, and turns into a moving wall. In modeling this stage of wave motion, the "shallow water" approximation again becomes inapplicable.

Under certain conditions, a tsunami can evidently appear not in the form of a wave train but in the form of a solitary wave (soliton).

It is very difficult to predict the time of tsunami arrival (and the height of the waves) at certain parts of the shore. The point is that little is known about the manner in which the height of a tsunami wave in the first kilometers of its path and the velocity of wave propagation along different paths change depending on the relief of the ocean bottom. And it is extremely difficult to predict the behavior of a tsunami directly at shores having a complex shape, bays, and inlets.

The fact that tsunami are nonlinear waves also considerably hampers reliable prognoses [2].

In the present paper we used the traditional "shallow water" approximation to solve problems of the behavior of gravitational waves. However, as will be shown in what follows, the range of its applicability is rather limited in modeling complex wave phenomena in ocean waters.

Simplification of the System of Equations in the "Shallow Water" Approximation. The "shallow water" approximation means that the amplitude of the wave is much smaller than the depth of the basin \( \varepsilon = a/h << 1 \) and the wavelength is much larger than the depth \( \delta = h/l << 1 \) [5].

Since we are considering long waves, we can neglect the total acceleration along the \( OZ \) axis in the equation

\[
\frac{\partial u_x}{\partial t} + (u_x, v) \frac{\partial u_x}{\partial y} + \frac{1}{\rho_0} \nabla P = -g y \tag{1}
\]

and can set

\[
\frac{du_z}{dt} = \frac{\partial u_z}{\partial t} + u_x \frac{\partial u_z}{\partial x} + u_z \frac{\partial u_z}{\partial z} = 0 .
\]

Then

\[
\frac{1}{\rho_0} \frac{\partial P}{\partial z} = -g . \tag{2}
\]

After integration over \( z \) with account for the boundary conditions we have

\[
P = P_0 + \rho_0 g (z^s - z) . \tag{3}
\]

We determine the derivative of the pressure entering the equation of motion

\[
\frac{\partial P}{\partial x} = \frac{\partial P_0}{\partial x} + \rho_0 g \frac{\partial z^s}{\partial x} . \tag{4}
\]

The right-hand side of (1) does not depend on the vertical coordinate in all changes of the liquid flow and the shape of the free surface. The pressure is eliminated from the equation of the dynamics using equality (4). Now the velocities are independent of the depth. Therefore, the derivative with respect to \( z \) disappears in the equation and, consequently, the vertical velocity no longer affects the horizontal flow. Thus, we have