NUMERICAL MODELING OF GRAVITATIONAL WAVES IN THE OCEAN BY THE DOUBLE-LAYER POTENTIAL METHOD

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The problem of wave structure formation on a water surface for a prescribed initial disturbance is considered. A method of modeling gravitational waves is suggested in which the velocity potential of the liquid is sought in the form of a double-layer potential. The method suggested is tested by solving the problem of wave propagation on shallow water. Calculations of formation of a wave train from the initial disturbance are made in the case of periodic motion in a two-dimensional model (the problem of Stokes waves). The results of the calculations have demonstrated the high effectiveness of the suggested method in solving problems in which it is necessary to determine a wave profile with good accuracy.

Analysis of wave propagation on water is a classical problem [1], and interest in it is dictated by the need to answer particular questions in solving a variety of applied problems.

The primary physical factors that determine the existence of waves on water are the gravitational forces and the surface tension [2]. In the present work we are concerned with gravitational waves.

The main conditions that exert an influence on wave propagation are the relations between the wavelength, amplitude, and depth of the liquid. They correspond to deep and shallow water. In the phenomenon of wave propagation these notions have a relative meaning and are compared to the wavelength. Deep water means bodies of water where the depth exceeds the wavelength, and shallow water means ones where it is less than the wavelength. Hence it follows that for a complex linear wave consisting of a number of sine waves with substantially different wavelengths a body of water can turn out to be both deep and shallow at the same time. In good agreement with observations, theory shows that water can be considered to be deep when the depth of the body of water exceeds approximately half the wavelength but water is shallow if the depth of the body of water is approximately ten times smaller than the wavelength.

The character of wave propagation on deep and shallow water is fundamentally different, which is attributable to the fact that wave motion quickly attenuates with increasing depth. The attenuation follows an exponential law, and already at a depth equal to half the wavelength the amplitude of displacement of water particles in the vertical direction is a factor of 23 smaller than on the water surface, and at a depth equal to a whole wavelength it is even more than 500 times smaller [3]. As a result, when the depth of a body of water is larger than a wavelength, the disturbance already scarcely reaches the bottom. In the case of shallow water, the entire body of liquid from the surface to the bottom is involved in the disturbance.

Rather short-lived sources of emergence of waves can serve as factors of excitation of long waves. These are earthquakes, eruptions of coastal and underwater volcanos, and underwater blasts. In the ocean, waves generated by an underwater shock often have a great length and are propagated with a high velocity. These waves, tsunami, often carry huge energy and when the waves break down on coasts they cause disastrous destruction [4].

Under initial conditions of a certain type tsunami evidently appears in the form of a solitary wave (a soliton) rather than in the form of a wave train. A soliton on water was observed by Russell in 1834 during barge tests on a canal. Stokes and Airy, contemporaries of Russell, regarded negatively the results of observations of this wave, but later Boussinesq (1872) and Rayleigh (1876) confirmed the possibility of Russell's observation of a solitary wave.
wave on water. In 1895 Korteveg and de Vries derived an equation based on the "shallow water" approximation [2] that corresponded to the Russell experiments.

Numerous works in which wave processes on water are modeled use, in essence, the "shallow water" approximation, i.e., an approximation in which the wave amplitude is much less than the depth of the body of water \( \varepsilon = a/h \ll 1 \) and the wavelength is much greater than the depth \( \delta = h/l \ll 1 \).

The present work suggests a calculation method and provides results of calculation of wave structure formation from an initial disturbance that are not based on the "shallow water" approximation.

**Formulation of the Problem.** Let an initial disturbance for which the "shallow water" approximations are violated at the initial moment be created in a body of water. Assume that at \( t = 0 \) all the characteristics of the disturbed region are known, in particular, the surface profile is known. In this model, the bottom relief is assumed to be horizontal [5-7].

This two-dimensional problem will be considered in a Cartesian coordinate system (the \( x \) axis is along the horizontal, the \( z \) axis is along the vertical). The problem is to determine the evolution in time of the liquid surface profile with allowance for gravitational forces.

Since water is practically incompressible, the continuity equation can be written in the form

\[
\text{div} \, \vec{u} = 0. \tag{1}
\]

By assuming that the liquid motion is a vortex-free flow, we can consider one scalar function \( \vec{u} = \nabla \varphi(x, z, t) \) instead of the two components of the vector function \( \vec{u} \) and one equation for the potential

\[
\Delta \varphi = 0. \tag{2}
\]

instead of two equations for the velocity components.

The equation of motion in the gravitational field has the form

\[
\frac{\partial \vec{u}}{\partial t} + (\vec{u}, \nabla) \vec{u} + \frac{1}{\rho_0} \nabla P = -g\vec{z}. \tag{3}
\]

Equation (3) can be integrated over \( z \) and represented as

\[
\frac{\partial \varphi}{\partial t} + \frac{1}{2} (\nabla \varphi)^2 + \frac{P - P_0}{\rho_0} + gz = 0. \tag{4}
\]

If we solve the Laplace equation for the velocity potential, then we can, in principle, find the pressure in the liquid from Eq. (4).

We now dwell on the boundary conditions of the problem under consideration. The equation of motion of the upper part of the boundary is of the form

\[
\frac{dz}{dt} = (\vec{u})_z = \frac{\partial \varphi}{\partial z}. \tag{5}
\]

After a standard substitution we write condition (5) as

\[
\frac{\partial z}{\partial t} + u_x \frac{\partial z}{\partial x} = u_z
\]

or

\[
\frac{\partial z}{\partial t} + \frac{\partial \varphi}{\partial x} \frac{\partial z}{\partial x} = \frac{\partial \varphi}{\partial z}. \tag{6}
\]

Assume that on the lower part of the boundary the following condition is fulfilled: