PERIODIC POINTS AND ROTATION NUMBERS FOR AREA PRESERVING Diffeomorphisms of the Plane

by John Franks

Abstract. — Let $f$ be an orientation preserving diffeomorphism of $\mathbb{R}^2$ which preserves area. We prove the existence of infinitely many periodic points with distinct rotation numbers around a fixed point of $f$, provided only that $f$ has two fixed points whose infinitesimal rotation numbers are not both 0.

We also show that if a fixed point $z$ of $f$ is enclosed in a "simple heteroclinic cycle" and has a non-zero infinitesimal rotation number $r$, then for every non-zero rational number $p/q$ in an interval with endpoints 0 and $r$, there is a periodic orbit inside the heteroclinic cycle with rotation number $p/q$ around $z$.

In this paper we investigate area preserving diffeomorphisms of $\mathbb{R}^2$ and the existence of periodic points with prescribed rotation number around a given fixed point. A motivating question for this investigation deals with a diffeomorphism $f: \mathbb{R}^2 \to \mathbb{R}^2$ which has two hyperbolic fixed points $P_1, P_2$, with a double saddle connection and an elliptic fixed point between the saddle connections (see Fig. 1a). The classical fixed point theorem of Poincaré and Birkhoff can be used to show that in this case for each rational $p/q$ between 0 and the infinitesimal rotation at $z$ there is a periodic orbit with rotation number $p/q$ around $z$ which lies inside the disk bounded by the saddle connections. This is done by "blowing up" the point $z$ to obtain a homeomorphism of the annulus bounded by the saddle connections and the blow up of $z$, and applying the theorem of Poincaré and Birkhoff to this annulus homeomorphism.

Fig. 1a
If the heteroclinic connections between the points \( p_1 \) and \( p_2 \) are more complicated (see Fig. 1b) this approach does not work, because there may be no invariant disk of finite area containing \( z \), but one can ask if the result is still true. With rather modest assumptions on the hyperbolic points in this more complicated case (see the definition of "simple heteroclinic cycle" in (3.2)), we show that it remains true that there is a periodic point with rotation number \( p/q \). This result is Theorem (3.4) below.

In § 2 we consider an even more general setting: any area preserving diffeomorphism \( f: \mathbb{R}^2 \to \mathbb{R}^2 \) which preserves orientation and has two fixed points \( z_0 \) and \( z_1 \). In (2.2) and (2.5) we show there are intervals with the property that for any non-zero \( p/q \) in their interior, there is a periodic point \( x \) such that \( p/q \) is the difference of the rotation numbers of \( x \) around \( z_1 \) and \( z_0 \). Such an interval can be chosen with endpoints the infinitesimal rotation number of \( z_1 \) and minus the infinitesimal rotation number of \( z_0 \) or with endpoints 0 and one of these numbers. In any case, unless the infinitesimal rotation numbers of \( z_0 \) and \( z_1 \) are both 0, there are infinitely many periodic orbits with distinct rotation numbers about at least one of the two fixed points.

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1. Background and definitions

We are interested in investigating the existence of periodic orbits for area preserving diffeomorphisms of \( \mathbb{R}^2 \) and measuring their rotation around a given fixed point. We begin by recalling the definition of rotation number for a homeomorphism of the annulus. Suppose \( f: \mathcal{B} \to \mathcal{B} \) is a homeomorphism of the annulus \( \mathcal{B} \) which is homotopic to the identity map (we consider \( \mathcal{B} = \mathbb{T}^1 \times I \), where \( I \) is \([0, 1] \), \((0, 1) \), or \([0, \infty) \)). Let...