Semi-classical limit of relativistic quantum mechanics

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MS received 26 March 2004; revised 5 January 2005; accepted 19 January 2005

Abstract. It is shown that the semi-classical limit of solutions to the Klein–Gordon equation gives the particle probability density that is in direct proportion to the inverse of the particle velocity. It is also shown that in the case of the Dirac equation a different result is obtained.

Keywords. Quasi-classical limit; Schroedinger equation; Klein–Gordon equation; Dirac equation.

PACS Nos 03.65.Pm; 03.65.Sq

1. Introduction

The semi-classical limit of the relativistic quantum mechanics can be introduced in the same way as the semi-classical limit of solutions to the Schroedinger equation. The purpose of this report is to consider one-dimensional stationary states and show the properties of the semi-classical limit of solutions to the Klein–Gordon and Dirac equations.

2. Partitioning the Klein–Gordon equation

Substitution

\[ \psi = R \exp(\frac{i\theta}{\hbar}), \]

where \( R \) and \( \theta \) are real, is used in the Klein–Gordon equation

\[ \left( i\hbar \frac{\partial}{\partial t} - e\phi \right)^2 \psi = c^2 \left( -i\hbar \nabla - \frac{e}{c} A \right)^2 \psi + m_0^2 c^4 \psi. \]
In this way eq. (2) is partitioned into equations

\[ c^2 \left( \nabla^2 \theta - \frac{e}{c^2} A \right)^2 - \left( \frac{\partial \theta}{\partial t} + e\phi \right)^2 + m_0^2 c^4 - c^2 \hbar^2 \frac{1}{R} \Delta R + \hbar^2 \frac{1}{R} \frac{\partial^2 R}{\partial t^2} = 0 \]

(3)

and

\[ c^2 R \Delta \theta - R \frac{\partial^2 \theta}{\partial t^2} + 2 e^2 \nabla R \cdot \nabla \theta - 2 \frac{\partial R}{\partial t} \frac{\partial \theta}{\partial t} - 2 e \frac{\partial R}{\partial t} = 0. \]

(4)

Equation (3) is the relativistic Hamilton–Jacobi equation with two additional quantum terms. Equation (4) is a relativistic continuity equation.

For a stationary state substitution (1) can be also written as

\[ \psi = R \exp(-iEt/\hbar) \exp(i\theta^*/\hbar), \]

(5)

where \( \theta = -Et + \theta^* \). In substitution (5) constant \( E \) and functions \( \theta^* \) and \( R \) are real and do not depend on time. This allows equation

\[ -\frac{\partial \theta}{\partial t} = E. \]

(6)

3. Semi-classical limit of the Klein–Gordon equation – Unbound states

Wave function (5) can describe both unbound and bound stationary states. In the case of unbound stationary states the energy of the particle \( E \) and the external potential \( \phi \) satisfy condition \( E - e\phi > 0 \), and \( R \) and \( \theta^* \) are non-sinusoidal functions of \( x \). The fact that \( R \) does not depend on time, condition (6), condition \( A = 0 \) and

\[ \hbar \to 0 \]

applied in (3) and (4) written for one dimension give equations

\[ -(E - e\phi)^2 + c^2 \left( \frac{\partial \theta^*}{\partial x} \right)^2 + m_0^2 c^4 = 0 \]

(8)

and

\[ R \frac{\partial^2 \theta^*}{\partial x^2} + 2 \frac{\partial R}{\partial x} \frac{\partial \theta^*}{\partial x} = 0. \]

(9)

Equation (9) can be rearranged as

\[ 2 \frac{\partial R}{R} = -\frac{\partial (\partial \theta^*/\partial x)}{\partial \theta^*/\partial x} \]

(10)

and solved for \( R \). This gives formula