Kinetic Model of Plasma Columns.

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Plasma structures in the form of cylindrical columns are often recurrent in theoretical and experimental plasma physics; a simple kinetic model for such structures is investigated here (its plane equivalent having been studied in detail recently by Lam (1)).

We introduce cylindrical co-ordinates $q, \theta, z$, and assume that all physical quantities depend on $\theta$ only. We assume furthermore that the electric field has only a $q$-component, and the magnetic field only $\theta$- and $z$-components. Under such symmetry conditions the (steady-state) equations for the vector and electric potentials read (in rationalized units, with $\mu$ and $\kappa$ as constants of the vacuum)

\begin{align}
\frac{1}{\mu} \frac{d}{dq} \left[ \frac{1}{\epsilon} \frac{d(\varphi A_q)}{dq} \right] &= -j_\theta = -(j_{q+} + j_{q-}), \\
\frac{1}{q} \frac{d}{d\theta} \left( q^2 \frac{dA_q}{d\theta} \right) &= -j_z = -(j_{q+} + j_{z-}), \\
\kappa \frac{1}{q} \frac{d}{d\theta} (q^2 \frac{d\varphi}{d\theta}) &= -q = -(q_+ + q_-).
\end{align}

Ion and electron charge and current densities in the above equations can be explicitly computed from the solutions of two Vlasov equations for the two species of particles,

\begin{align}
v_e \frac{\partial f_{e\pm}}{\partial q} \pm \frac{e}{m_e} \left[ (v_0 B_z - v_\theta B_\theta + E_\theta) \frac{\partial f_{e\pm}}{\partial v_\theta} - v_q B_z \frac{\partial f_{e\pm}}{\partial v_q} - v_q B_\theta \frac{\partial f_{e\pm}}{\partial v_q} \right] &= 0,
\end{align}

where

\begin{align}
B_\theta &= -\frac{1}{\mu} \frac{dA_z}{dq}, \\
B_z &= \frac{1}{e} \frac{d(\varphi A_q)}{dq}, \\
E_\theta &= -\frac{1}{q} \frac{d\varphi}{dq}.
\end{align}

(1) S. H. Lam: private communication.
Any function of the constants of the motion

\[ \epsilon = \frac{1}{2} m_\pm (v_\epsilon^2 + v_\theta^2 + v_z^2) \pm \epsilon \rho, \quad p_\theta = q(m_\pm v_\theta \pm \epsilon A_\phi), \quad p_z = m_\pm v_z \pm \epsilon A_z \]

is a solution of (2), as can be simply verified. If all three constants of the motion appear as arguments of the distribution functions then in general the current and the magnetic field have both a \( \theta \)- and a \( z \)-component (helical geometry). If instead the distribution functions depend on \( \epsilon \) and \( p_z \) but not on \( p_\theta \), the current is in the \( z \)-direction and the magnetic field in the \( \theta \)-direction (linear-pinch geometry). Finally, if the distribution functions depend on \( \epsilon \) and \( p_\theta \) but not on \( p_z \), the current is in the \( \theta \)-direction and the magnetic field in the \( z \)-direction (theta-pinch geometry). We discuss here specific examples of the last two cases.

Let us first consider distribution functions of the following form (linear-pinch geometry):

\[ f_\pm(\epsilon, p_\theta, p_z) = N \left( \frac{m_\pm}{2\pi K T_\pm} \right)^{\frac{3}{2}} (1 + \eta_\pm)^{\frac{1}{2}} \exp \left[ -\eta_\pm \frac{p_z^2}{2m_\pm K T_\pm} - \frac{\epsilon}{K T_\pm} \right], \]

where \( N \) is a reference-number density, \( K \) is Boltzmann’s constant, \( T_\pm \) are reference temperatures for ions and electrons, and \( m_\pm \) are the particle masses. The \( \eta_\pm \) instead are dimensionless free parameters that have been introduced for flexibility; here we consider \( \eta_\pm > 0 \). Computing charge and current densities from (5), one obtains first of all \( j_{\theta\pm} = 0 \) (due to the \( f_\pm \) not depending on \( p_\theta \)), hence the first of eqs. (1) can be trivially satisfied by taking \( A_\theta = 0 \). Furthermore, since the ion and electron charge densities are of the form

\[ q_\pm = \pm e N \exp \left[ -\eta_\pm \frac{e^2 A_z^2}{1 + \eta_\pm 2m_\pm K T_\pm} \pm \frac{\epsilon}{K T_\pm} \right], \]

the third of eqs. (1) can also be trivially satisfied by taking \( \varphi = 0 \), provided the \( \eta_\pm \) are subject to the condition

\[ m_+ T_+(1 + \eta_+) / \eta_+ = m_- T_-(1 + \eta_-) / \eta_- = W, \]

which we require here. Thus we are left with the second of eqs. (1), with the ion and electron current given by

\[ j_{\pm \pm} = -\frac{e^2 N T_{\pm \pm}}{W} A_z \exp \left[ -\frac{e^2 A_z^2}{2KW} \right]. \]

Notice that the two currents are in the same ratio as the temperatures.

Requiring regularity of all physical quantities at the origin, one obtains the following asymptotic behaviour for small \( \varphi \):

\[ A_z \sim A_{z0} \left( 1 + \frac{1}{4} \alpha^2 \right), \quad \alpha = \frac{\mu e^2 N (T_+ + T_-)}{W} \exp \left[ -\frac{e^2 A_{z0}^2}{2KW} \right], \]

where clearly \( A_{z0} \) is the value of \( A_z \) at the origin. Equations (9) can be used to obtain