The Formulation of Constitutive Equations in Continuum Relativistic Physics.

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(ricevuto il 10 Giugno 1969; manoscritto revisionato ricevuto il 18 Ottobre 1969)

Summary. — In this paper we examine the restrictions in the form of constitutive functions of relativistic continuum physics resulting from the requirement of form-invariance under the Lorentz group. These functions are assumed to depend on the deformation gradient and a number of vectors. It is shown that the restricted forms are the relativistic extensions of similar expressions of classical continuum physics derived by RIVLIN.

Introduction.

Several physical properties of materials can be expressed by tensorial relations. In classical physics such tensors are expressed by components in a 3-dimensional coordinate system of the classical space, while the time enters into the constitutive equations as an independent parameter. The restriction imposed on the constitutive equations by the principle of «material objectivity» (1) is expressed as form-invariance under time-dependent rotation of the physical system (transformation of Euclidean frames). Additional restrictions on constitutive equations due to symmetries, which the material possesses in its reference state, are expressed as form-invariance under a group of transformations which is a subgroup of the full orthogonal group.

The general theory of constitutive equations in classical continuum physics has been developed by RIVLIN (2), and PIPKIN and RIVLIN (3) for tensorial equations of any order in Cartesian form.

In the theory of physics based on special relativity the principle of objectivity is stated as follows: the constitutive equation must be form-invariant under a homogeneous one-parameter proper Lorentz transformation.

It is known that, in relativistic physics, time and space co-ordinates cannot be considered as independently transformed parameters. A convenient geometric method is to describe the motion of a material continuum by a congruence of world lines in a 4-dimensional Minkowskian space-time. The motion of a material point is described in a Lorentz frame which is fixed with respect to an observer by three spatial and one temporal co-ordinates.

Therefore all tensorial quantities which characterize various material properties must be expressed as four-tensors. The restrictions due to material objectivity are equivalent to form-invariance in Lorentz transformation of 4-dimensional space-time. On the other hand the restrictions imposed by material symmetries are studied in the 3-dimensional space of material co-ordinates in the reference (rest) state as form-invariance under orthogonal transformations.

In this paper we extend the theory of ref. (1.3) into relativistic physics by examining the restrictions in the 4-tensor constitutive equations due to Lorentz covariance and the integrity basis for invariance under the 3-dimensional orthogonal group which represents the material symmetries.

1. – Notation and kinematic preliminaries.

In the following, minuscule Greek indices will denote components in 4-dimensional space-time while majuscule Latin indices will denote components in the 3-dimensional material co-ordinate system.

Let \( X^\alpha \) be the co-ordinates of a material point in a Cartesian system in the reference state. The co-ordinates of the world line of the same material point are denoted by \( x^\alpha = \{ x^1, x^2, x^3, x^4 = ct \} \), where \( x^1, x^2, x^3 \) the spatial Cartesian co-ordinates and \( t \) the time measured with respect to a fixed Lorentz frame \( x \) while \( c \) denotes the speed of light in a vacuum. The indices 1, 2, 3, correspond to spatial co-ordinates while the index 4 to the temporal co-ordinate. The geometry of space-time is expressed by the quadratic form

\[
(1.1) \quad ds = (-g_{\alpha\beta}dx^\alpha dx^\beta)^{1/2},
\]

where \( g_{\alpha\beta} \) is the (covariant) metric having canonical form

\[
(1.2) \quad g_{\alpha\beta} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]