A Mathematical Inequality
Related to the $H$ Theorem Based on Coarse Graining (*)

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Summary. — The $H$ theorem based on coarse graining may be considered as a particular case of a more general mathematical theorem. Since both theorems are very similar caution is necessary not to confuse them. The mathematical theorem may possibly have some other physical applications.

1. - Introduction.

Since BOLTZMANN presented in 1872 the famous $H$ theorem to discuss the approach to equilibrium there have been reported many oppositions, arguments and modifications. We have now roughly the following four versions of the $H$ theorem. All of these try to show the entropy increase towards statistical equilibrium but differ in their contents from each other:

a) Dynamical approach:
   1) $H$ theorem based on Boltzmann's equation.
   2) $H$ theorem based on reduced Liouville's equation and the master equation.

b) Statistical approach:
   3) $H$ theorem based on the canonical distribution.
   4) $H$ theorem based on a coarse grained distribution.

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It seems unnecessary to explain these versions of the $H$ theorem since they are discussed in current books and literature widely. It is the purpose of this note to point out that the last $H$ theorem based on a coarse-grained distribution function is a particular case of a more general mathematical theorem which differs slightly from Theorem I by Gibbs discussed in his famous book in not specifying the standard distribution (in Gibbs' notation the index $\eta$) as a function of the energy.

We shall give in the next Section the mathematical theorem which is very similar yet different from the $H$ theorem of our concern. In Sect. 3 we shall remark on the $H$ theorem and point out the difference between the two theorems, and in Sect. 4 we shall give a simple example for further elucidation.

2. – Mathematical inequality.

We shall omit all detailed discussions concerning the $H$ theorem and present a mathematical theorem which includes the $H$ theorem in a special case.

Theorem. For two distribution functions $f(x)$ and $\tilde{f}(x)$ in the $\Gamma$ space satisfying the condition

\begin{equation}
\int_{\Gamma} f(x) \, dx = \int_{\Gamma} \tilde{f}(x) \, dx ,
\end{equation}

define

\begin{align*}
H &= \int_{\Gamma} f \log f \, dx , \\
\tilde{H} &= \int_{\Gamma} \tilde{f} \log \tilde{f} \, dx .
\end{align*}

Then, the following inequality holds:

\begin{equation}
H \geq \tilde{H} .
\end{equation}

The proof of this inequality may be given following the well-known Gibbs,

\begin{itemize}
\item[(2)] R. C. Tolman: *The Principles of Statistical Mechanics* (Oxford, 1938);
\end{itemize}