Local noncuppability in $R/M$

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Abstract Given any $[c], [a], [d] \in R/M$ such that $[d] \leq [a] \leq [c]$, $[a]$ is locally noncuppable between $[c]$ and $[d]$ if $[d] < [a] \leq [c]$ and $[a] \lor [b] < [c]$ for any $[b] \in R/M$ such that $[d] \leq [b] < [c]$. It will be shown that given any nonzero $[c] \in R/M$, there are $[a], [d] \in R/M$ such that $[d] < [a] \leq [c]$ and $[a]$ is locally noncuppable between $[c]$ and $[d]$.

Keywords: recursively enumerable degree, cappable, semilattice.

1 Introduction

The local noncuppable properties in $R/M$ are studied in this paper. Let $[c], [a], [d]$ be elements in $R/M$ such that $[d] \leq [a] \leq [c]$. We say that $[a]$ is locally noncuppable between $[c]$ and $[d]$ if $[d] < [a] \leq [c]$ and $[a] \lor [b] < [c]$ for any $[b] \in R/M$ such that $[d] \leq [b] < [c]$. Ambos-Spies et al. proved that $M$, the set of all the cappable r.e. degrees, is an ideal in $R$; $NC$, the set of all the noncappable r.e. degrees, is a filter in $R$, and $NC$ is equal to $PS$, the set of all the promptly simple degrees. Hence we have a quotient of $R$ modulo $M$, denoted by $R/M$. Schwarz proved the downward density theorem in $R/M$. Ambos-Spies (Jockusch) commented that the downward density theorem in $R/M$ follows directly from the Robinson’s splitting theorem and the fact that $NC$ is equal to $LCU$, the set of all the r.e. degrees which cup to $0'$ by some low r.e. degree. Lempp communicated the following open problem listed in Slaman’s “open problems in recursion theory”:

Open problem. Does the Shoenfield conjecture hold in $R/M$?

Although this problem has been refuted by Yi, and independently by Sui, who proved that there are $[a], [d] \in R/M$ such that $[a]$ is locally noncuppable between $[0']$ and $[d]$, it also can be seen that the structure $R/M$ has many wonderful properties and is quite different from that of the structure $R$. One can easily see that there is no minimal pair in $R/M$. Sui and Zhang proved that the Shoenfield cupping conjecture holds in $R/M$, i.e., for any $[a], [b] \in R/M$ such that $[a] < [a] < [b]$, there is an r.e. degree $c$ such that $[c] < [b]$ and $[b] = [a] \lor [c]$. We shall go on with our study of the structure $R/M$ in this paper and prove that for any nonzero $[c] \in R/M$, there are $[a], [d] \in R/M$ such that $[d] < [a] \leq [c]$ and $[a]$ is locally noncuppable between $[c]$ and $[d]$.

Our notation is standard with a minor change. A number $x$ is unused at stage $s$ if $x \geq s$ is greater than any used number so far in the construction. If the oracle is a join of two sets we
assume that the use is computed on the two sets separately, i.e., \( \Gamma((A \oplus B)[(\gamma(x) + 1); x]) = \Gamma(A[(\gamma(x) + 1) \oplus B[(\gamma(x) + 1); x]), \) where \( \gamma(x) \) is the use of \( \Gamma(A \oplus B; x) \). All the used functions are assumed to be increasing in argument and nondecreasing in the stages.

Given any recursive functional \( \Phi_e(A) \) with oracle \( A \), we shall use \( \varphi_e,s(x) \) to denote \( u(A; e, x, s) \). During our construction, the following lemma will be used:

**Slowdown lemma** \(^7\). Let \( \{G_{e,s}\} \) be a strong array of finite sets such that \( G_{e,s} \subseteq G_{e,s} \) and \( G_e = \bigcup G_{e,s} \). Then there is a recursive function \( h \) such that for all \( e \) and \( s \), \( W_{h(e)} = G_e \) and \( W_{h(e),s} = G_{e,s} \).

2 The main theorem and its requirements

**Theorem 2.1.** Given any nonzero \( [c] \in R/M \) there are \( [a], [d] \in R/M \) such that \( [d] < [a] < [c] \) and \( [a] \) is locally noncuppable between \( [c] \) and \( [d] \).

To prove Theorem 2.1, let \( C \) be a promptly simple r.e. set of degree \( c \). Then by promptly simple degree theorem in ref.\(^1\), there is a recursive function \( f, \) such that for every \( e, \)

\[
\| W_e \| = \infty \rightarrow \exists x \ni \exists s(x \in W_{e,s} \ni & G_f(x) \ni x \neq C_f(x) \ni x).
\]

We shall construct r.e. sets \( A, D, D^0, D^1, E^0, E^1, D_e, E_e, \) a \( \Delta_0 \)-set \( B \) and recursive functionals \( \Gamma^0, \Gamma^1, \Gamma_e \) such that \( A \oplus D = \Gamma^0(C \oplus D^0), B = \Gamma^1(C \oplus D^1), \) and for every \( e, i \in \omega, k \in \{0,1\} \), the following requirements are satisfied.

\[
\begin{align*}
\mathcal{P}_e & : E_e \neq \{e\}, \\
\mathcal{M}_e & : \| e \|^D = \{e\}^{k, e} = f_e^k \text{ total } \rightarrow \mathcal{M}_e \subseteq \mathcal{M}, \\
\mathcal{R}_e & : A = \Phi_e(D \oplus V_e) \rightarrow \deg_T(V_e) \in \text{NC}, \\
\mathcal{N}_e & : B = \Psi_e(A \oplus D \oplus U_e) \rightarrow C = \Gamma_e(D \oplus U_e \oplus D_e) \land (\forall i < \omega)(\mathcal{P}_{e,i} \land \mathcal{M}_{e,i}),
\end{align*}
\]

where

\[
\begin{align*}
\mathcal{P}_{e,i} & : E_e \neq \{i\}, \\
\mathcal{M}_{e,i} & : \{i\}^D = \{i\}^{D_e} = f_{e,i} \text{ total } \rightarrow \mathcal{M}_{e,i} \subseteq \mathcal{M},
\end{align*}
\]

where \( \{(V_e, U_e, \Phi_e, \Psi_e)\} \) is a standard enumeration of all such quadruples \((V, U, \Phi, \Psi)\) that \( V, U \) are r.e. sets and \( \Phi, \Psi \) are recursive functionals. In the construction, we will decompose \( \mathcal{R}_e \) into infinitely many sub-requirements; that is, for very \( i < \omega \):

\[
\mathcal{R}_{e,i} : A = \Phi_e(D \oplus V_e) \land \| W_i \| = \infty \rightarrow \exists x \ni \exists s(x \in W_{i,s} \ni & V_e,s \ni x \neq V_{e,p_e(s)} \ni x),
\]

where \( p_e \) is a partially recursive function to be defined in the construction to show the prompt simplicity of \( V_e \), and \( \| W_i \| \) is a standard enumeration of all the recursively enumerable sets.

Theorem 2.1 holds if all the above conditions are satisfied. Let \( a = \deg_T(A \oplus D) \). Then the requirements \( \mathcal{P}_e^k \) and \( \mathcal{M}_e^s \) will guarantee that \( \deg_T(D^k) \in M \). By the requirements \( \mathcal{R}_e \) and the definition of \( \Gamma^0 \), we have \( [a] \leq [d] \) and \( [a] \leq [c] \), thus \( [d] < [a] < [c] \). Let \( [b] \) be any element in \( R/M \) such that \( [d] \leq [b] < [c] \). Then the requirements \( \mathcal{N}_e \) together with all \( \mathcal{P}_{e,i} \) and \( \mathcal{M}_{e,i} \) will guarantee that \( [a] \cup [d] < [c] \).

3 The priority tree and the basic modules

3.1 The priority tree

The priority tree \( T \) is the complete binary tree. We define an order \( < \) on \( T \) as follows: for any \( \alpha, \beta \in T \),