Geometry from Quantum Mechanics (*).

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Summary. — A model is proposed in which the standard commutation relations of quantum mechanics are realized in phase space by representing both position and momentum operators as absolute covariant derivatives, whose commutator, taken over from QM, is the curvature of phase space. When \( \hbar \to 0 \), the \( q \) and \( p \) subspaces become disconnected and classical mechanics is recovered. Bohr-Sommerfeld quantization rules for integer and half-integer values, nonrelativistic and relativistic QM with Klein-Gordon and Dirac equations follow without further assumptions. It appears that models of this sort are apt to describe particles with internal degrees of freedom and may lead to a unified description of QM and GR, somewhat in accordance with Born's reciprocity principle.

1. — Introduction.

I'll. The model. — Can quantum mechanics (QM) be expressed as pure geometry, as was the case with gravity and general relativity? Attempts in this direction are very numerous, many with a degree of geometrical sophistication well beyond this writer's ability. The main risk in building up « models » of the sort (we refrain from the word « theory ») is to find at the end a formalism more complicated than the familiar one of standard QM, not easily (if at all) falsifiable, with no compelling reason for physicists to like it better than QM as it is. The effort remains, nevertheless, a worthy one, because the

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conceptual basis of QM is still an object of heated debate and also because it may stimulate new outlooks which may prove helpful in the present trend towards «unification», or just in suggesting new computational techniques.

Well aware of these difficulties, we wish to propose here for discussion a declaredly naive model of quantization «in phase space»; it rests upon a single assumption, from which all else necessarily follows with familiar methods (variants are of course possible; some will be briefly mentioned but not discussed). We take it as a «dogma» here that the commutation relations between position and momentum operators in orthogonal co-ordinates for one-particle QM are a fact not to be disputed. QM imposes constraints that act only on $q_1, p_1$, in the particle's phase space. Can we represent both sets of operators $Q_1, P_1$ as covariant derivatives (we shall see that this requires actually an 8-bein formalism, which we call «quantum frame»), so that the commutators of these determine the curvature tensor of phase space, whose values one can then take over from standard QM!

The present model consists in assuming tentatively that this is possible and proceeding then to set up the simplest geometric structure that may permit to pursue this program. The necessary steps are the following. Knowledge of the curvature tensor (from QM: we may think of a free particle to fix ideas, though this will not be necessary) gives equations for the connection; these have many solutions (cf. ref. (1) for some examples), of which only the simplest one is considered here for use. The connection is thus the fundamental element in our game: assigning it is the same as assigning the physics; our assumption is better stated by the last sentence. Of the connection it is required that it yields the wanted commutation relations for correspondence with QM (and any number of additional requirements, such as necessary to secure integrability conditions, etc., may as well be imposed on it for the model to make sense; this will be relevant especially if GR shall have to be made a part of the model, but is not done here, to keep things to bare essentials). «Minimal coupling» is thus part of our assumption; EM fields, etc. are naturally included in this way.

The connection can be split into a metric and a gauge part; this splitting is not unique in general, the two parts being to some extent interchangeable; not surprisingly, because the metric tensor itself, which is computed from the metric part of the connection, turns out to be, in general, Hermitian (for a simple computation cf. ref. (1)). It is apparent at this point that the connection can be chosen so as to give GR in both $q$ and $p$ subspaces, besides the standard QM relations between them: «unification» appears as a natural aim in this model, which is at this point strongly reminiscent of Born's reciprocity principle (ref. (1)). Although this seems to be a legitimate expectation, the

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