The Probability Current for Scalar Particle in General Relativity (*).

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Summary. — We discuss the generalized formulation for the probability distribution for scalar particle in general relativity. This is tested by the correspondence principle. We show in the high-frequency short-wavelength limit that the generalized quantities exhibit correct physical properties. However, the formulation suffers a normalization problem. There, for example, a wave function might not be simultaneously normalizable in all reference frames. These conclusions are demonstrated in the Schwarzschild geometry, using suitable metrics, for i) a stationary-wave solution near the event horizon and ii) a general normalizable wave packet solution of the scalar-field equation. In this paper we also establish the geodesic trajectory and the Heisenberg uncertainty principle for a wave packet in curved space-time.

1. - Introduction.

The scalar-field equation in general relativity has attracted considerable interests (1-6). It seems, however, that no attention has been paid to examine the generalization of the associated probability current to the new domain. The probability formulation is fundamental for the quantum-

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mechanical description. Here we undertake this problem, making a particular application to the Schwarzschild geometry.

In flat space-time the interpretation of the probability current for scalar particle has a well-known difficulty \(^{(7,8)}\), \textit{i.e.}, for a general superposition of states with positive and negative energies, the probability density may take both positive and negative values. The present study in curved space-time shows that when we deal with superposition of only positive- or negative-energy states, we still get the consistent probabilistic interpretation, except, however, for a normalization problem.

As at present there is no access for experimental verification, we can at most rely on the correspondence principle to test if our generalization yields the correct intuitive result. This is being demonstrated in unperturbed Schwarzschild geometry, using suitable metrics for i) a stationary-wave solution near the event horizon and ii) a general normalizable wave packet solution of the scalar-field equation.

It is known that scalar-field perturbation of Schwarzschild geometry at the event horizon can drastically affect the background metric \(^{(9)}\). However, our idealization of unperturbed background geometry, particularly near the event horizon, should not affect the conclusion regarding the correct probabilistic interpretation. This point will become evident by recognizing that the formulation of the probability current does not make reference to the actual source of the background geometry, \textit{i.e.} to the Einstein field equation. Furthermore, there will result no ambiguity that might call for perturbation correction.

2. - The probability current.

The general-relativistic generalization to the Klein-Gordon equation for a free scalar particle reads
\begin{equation}
\Phi^{\mu}_{\nu} = - \frac{m_0^2}{\hbar^2} \Phi = 0 ;
\end{equation}
the metric signature is \(+2\). In eq. (2.1), \(\Phi\), \(m_0\) and \(\hbar\) are, respectively, the scalar field, the invariant mass of the particle and Planck's constant. The correct classical limit of eq. (2.1) will be discussed in the next Section.

The probability current density for a scalar particle in curved space-time may then be defined as \(^{(7,8)}\)
\begin{equation}
S_\mu = \frac{\hbar}{2im_0} \left( \Phi^* \frac{\partial}{\partial x^\mu} \Phi - \Phi \frac{\partial}{\partial x^\mu} \Phi^* \right).
\end{equation}