Remarks on the Brans-Dicke Theory of $N$ Dimensions (*).

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Summary. — A study of the Brans-Dicke theory in vacuum shows that one of the Brans-Dicke equations is represented in terms of the Riemann curvature tensor, the Weyl projective curvature tensor and its covariant derivative. Studying the case that the covariant derivative of the Weyl projective curvature tensor vanishes, we make a comment on a relation between the Brans-Dicke space-time and a space-time of constant curvature.

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1. – Introduction.

In vacuum the Brans-Dicke equation (1) can be derived from the requirement that the variations of the action

\begin{equation}
S = \int [\varphi R + \omega (\varphi_{\mu} \varphi^{\mu} / \varphi)] (-g)^{1/2} d^{N}x
\end{equation}

with respect to $\varphi$ and with respect to $g_{\mu\nu}$ vanish, where $\varphi$ is a positive scalar, $R$ is the scalar curvature and $\varphi_{\mu}$ is the differentiation of $\varphi$ with respect to $x^{\mu}$. $\omega$ is a real parameter. The field equation for $\Psi$ ($\Psi = -\ln \varphi$) is

\begin{equation}
\Psi^{\alpha}_{;\beta} - \frac{1}{2} \Psi^{\alpha}_{\lambda} \Psi^{\lambda} + (2\omega)^{-1} R = 0
\end{equation}

(*) To speed up publication, the author of this paper has agreed to not receive the proofs for correction.

where $\Psi_{i;\lambda}$ is the covariant derivative of $\Psi^i$ (where $\Psi^i = g^{ij} \Psi_j$);

\begin{equation}
R_{\mu\nu} = \Psi_{\mu;\nu} - (1 + \omega) \Psi^\mu \Psi_{\nu} + \frac{1}{2} g_{\mu\nu} (\Psi_{i;\lambda} - \Psi^i \Psi_{\lambda}) ,
\end{equation}

where $R_{\mu\nu}$ is the Ricci tensor; the trace of $R_{\mu\nu}$ is

\begin{equation}
R = (1 + \frac{1}{2} N) \Psi^\lambda \Psi_{\lambda} - (1 + \omega + \frac{1}{2} N) \Psi^i \Psi_{\lambda} .
\end{equation}

From (2) and (4) we obtain

\begin{equation}
(1 + 2 \omega + \frac{1}{2} N) (\Psi_{i;\lambda} - \Psi^i \Psi_{\lambda}) = 0 .
\end{equation}

If $1 + 2 \omega + \frac{1}{2} N$ is not zero (if it is zero, for $N = 4$, $\omega$ is typically $-\frac{3}{2}$), then

\begin{equation}
\Psi_{i;\lambda} - \Psi^i \Psi_{\lambda} = 0 ;
\end{equation}

the case $1 + 2 \omega + \frac{1}{2} N = 0$ will not be treated here (2). Using (6), we can rewrite (3) and (4) as follows:

\begin{equation}
R_{\mu\nu} = \Psi_{\mu;\nu} - (1 + \omega) \Psi^\mu \Psi_{\nu} ,
\end{equation}

\begin{equation}
\Psi_{i;\lambda} = \Psi^i \Psi_{\lambda} = - R/\omega .
\end{equation}

For later convenience, we write (7) in terms of $U$ which equals $(1 + \omega) \Psi$:

\begin{equation}
(1 + \omega) R_{\mu\nu} = U_{\mu;\nu} - U_{\mu} U_{\nu} .
\end{equation}

In the next section, by using (9), the Weyl projective curvature tensor and its covariant derivative, we will obtain a new formula which plays an important role in the later discussion.

2. - Formulation.

The Weyl projective curvature tensor $W^{\sigma}_{\mu\nu\lambda}$ which is hereafter called the Weyl tensor and is invariant under a projective change of affine connection will be defined as follows:

\begin{equation}
W^{\sigma}_{\mu\nu\lambda} = R^{\sigma}_{\mu\nu\lambda} + \frac{1}{N - 1} (\delta^{\sigma}_{\nu} R_{\mu\lambda} - \delta^{\sigma}_{\lambda} R_{\mu\nu}) .
\end{equation}