**Rigorous Lower Bounds to Critical Exponents for Ferromagnetic Ising Systems (*)**.

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(ricievuto il 23 Settembre 1975)

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**Summary.** — Rigorous lower bounds to the critical exponents $\gamma$, $\nu$ in ferromagnetic Ising systems are derived, using an idea of Glimm and Jaffe and some ideas and results of Fisher.

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1. **Introduction.**

Glimm and Jaffe (1) have recently proved that the critical indices $\eta$, $\nu$ and $\gamma$ (2) are bounded from below by their classical (Goldstone) values in the $(\varphi^4)_2$ quantum field theory, namely

1. $\eta > 0$,
2. $\nu > \frac{1}{2}$,
3. $\gamma > 1$.

One consequence of their paper is, among others, the inequality

4. $2\nu > \gamma$,
5. $2\nu > \alpha$.

Probably all the above field theory results hold also in statistical mechanics. In fact, one of them (3) is, in field theory, a corollary of an inequality which

(*) To speed up publication, the author of this paper has agreed to not receive the proofs for correction.

holds for ferromagnetic Ising systems (Lebowitz's inequality (1)), together with the so-called lattice approximation (4). However, most of the proofs given in field theory for (1)-(5) are specific to field theory: for instance, the simple inequality (1) is, in field theory, a simple consequence of the Källén-Lehmann representation in space-time dimensions higher than two (4), while in two dimensions it follows from a result of Simon (4). A proof of (1) in statistical mechanics exists however (and is trivial) only in two dimensions, if we define $\eta$ by (*)

$$\Gamma_r(r) - \Gamma^\infty_r \sim \frac{1}{r^{d-2+\eta}},$$

where $\Gamma_r$ is the (infinite volume) $r$-spin correlation function.

The proof of (3) in statistical mechanics according to the method of (2) also requires some additional argument, which we provide in sect. 2. In particular, two assumptions (A) and (B) of sect. 2), some applications of Griffiths' inequalities and a result of Fisher (6) are required. The r.h.s. of (3) is just the mean-field exponent (see, e.g., (4)), and we feel that the importance and interest of this result makes isolation of the necessary assumptions and a complete proof desirable, although the central idea is, of course, provided by (1). As a corollary of (3) we have, by Fisher's inequality (8)

$$v > \frac{1}{d},$$

$$0 < 2 - \eta < d,$$

the bound

where $v$ is the (dimension of the system), which is weaker than (2) and coincides with it only for $d = 2$. Of course, (2) would follow from (1), (3) and (7), but a proof of (1) is at present missing, as remarked previously.

2. – Result.

We consider a ferromagnetic Ising system. If $\Omega_N$ is a finite lattice region of $N$ spins in $Z^d$, we define its Hamiltonian by

$$H_\Omega (\{S_i\}_{i \in \Omega_N}) = - \sum_{p, q \in \Omega_N} J(p - q) S_p S_q,$$

(*) B. Simon: The $p(p)_1$ Euclidean (Quantum) Field Theory (Princeton, N. J., 1974).
(*) The notation $f(x) \sim x^a$ means $\lim_{x \to 0} (\log f(x)/\log x) = a$ for $x \in R_+^\ast, f(x) \in R_+$, as is standard (4).