Solution of a Nonlinear Sturm-Liouville Eigenvalue Problem and the Question of Motion Stability in Intersecting Storage Rings (*)(**).

P. A. Vuillermot

Department of Mathematics, Emory University - Atlanta, Ga. 30322

(ricevuto 1'8 Luglio 1981)

Nonlinear Hill's equations of the form

\[ z''(\Theta) + n(\Theta)z(\Theta) = F(\Theta; z(\Theta)) \]

frequently occur in the description of betatron oscillations in cyclic accelerators and storage rings (see (1) for the relevant background information and also (***) for a more recent account on the subject). In eq. (1), \( \Theta \) stands for the azimuth around the machine (of radius 1), \( n \) denotes a periodic function with (minimal) period \( T < 2\pi \), while \( F \) generally depends nonlinearly on \( z \) and also periodically on \( \Theta \); in this letter, we announce new results concerning the existence of Sturm-Liouville eigenfunctions and of periodic solutions for a class of equations of the form (1), namely

\[ z''(\Theta) + nz(\Theta) = \beta A(\Theta)F(z(\Theta)), \]

where \( n \) and \( \beta \) are real parameters. More specifically, we assume that \( A \) is a smooth periodic function defined on \( \mathbb{R} \) with minimal period \( T > 0 \); we, moreover, assume that \( A \) is uniformly bounded in \( \Theta \), namely that there exists \( K > 0 \) such that \( 0 < |A(\Theta)| < K \); finally we assume that \( F : \mathbb{R} \to \mathbb{R} \) is smooth and that its potential \( G \), with \( G' = F \), is subharmonic in the sense that the inequalities \( 0 < G(t) < \eta t^2 \) hold for each \( t \in \mathbb{R} \) for some \( \eta > 0 \) independent of \( t \). For \( \gamma, \gamma', \delta, \delta \in \mathbb{R} \) given, we then associate with (2) the

(*) Work supported in part by the Emory University Research Committee under Grant WOOD RES 4369 and in part under NSF-Grant INT-8024147.
(**) Lecture delivered at the Sixth International Conference on Mathematical Physics, Berlin, West Germany, August 1981.


Sturm-Liouville separated boundary conditions

\[
\begin{align*}
\gamma z(\Theta_1) - \gamma' z'(\Theta_1) &= 0, \\
\delta z(\Theta_2) + \delta' z'(\Theta_2) &= 0,
\end{align*}
\]

where \(\gamma^2 + \gamma'^2 = \delta^2 + \delta'^2 = 1\), with \(-\infty < \Theta_1 < \Theta_2 < +\infty\) fixed once and for all. For a given value of \(\beta\), we want to determine a function \(\beta \rightarrow n(\beta)\) (the eigenvalue) and nontrivial, classical (i.e., twice continuously differentiable) real-valued functions \(z^*(\Theta; \beta)\) (the eigenfunctions) such that

\[
- z^{**}(\Theta; \beta); \beta \Delta(\Theta) F(z^*(\Theta; \beta)) = n(\beta) z^*(\Theta; \beta)
\]

holds for each \(\Theta \in [\Theta_1, \Theta_2]\), along with the boundary conditions (3). In this direction, we first propose the following result (we have used the standard notation \(\text{sgn}(x)\) for the sign function defined by \(\text{sgn}(x) = +1\) for \(x > 0\) and \(\text{sgn}(x) = -1\) for \(x < 0\):)

Theorem 1. Let \(\mu > 0\) and \(\gamma \neq 0, \delta \neq 0\). If \(\gamma \neq 0\) and/or \(\delta \neq 0\), assume \(\text{sgn}(\gamma) = \text{sgn}(\gamma')\) and/or \(\text{sgn}(\delta) = \text{sgn}(\delta')\) and, along with the above assumptions on \(A\) and \(G\), that \(\beta A < 0\) on \([\Theta_1, \Theta_2]\). Then there exists a one-parameter family \(\{z^*_\mu(\Theta; \beta)\}_{\mu \in \mathbb{R}^*}\) of nontrivial classical solutions satisfying (3) and functions \(\beta \rightarrow n(\mu; \beta)\) such that

\[
z^{**}_\mu(\Theta; \beta) + n(\mu; \beta) z^*_\mu(\Theta; \beta) = \beta A(\Theta) F(z^*_\mu(\Theta; \beta))
\]

holds everywhere on \([\Theta_1, \Theta_2]\). Moreover, we have

\[
n(\mu; \beta) = \frac{1}{\mu} \left\{ \gamma (z^{**}_\mu(\Theta_1; \beta)) + \delta (z^{**}_\mu(\Theta_2; \beta)) + \int_{\Theta_1}^{\Theta_2} d\Theta \left\{ (z^{**}_\mu(\Theta; \beta))^2 + \beta A(\Theta) z^*_\mu(\Theta; \beta) F(z^*_\mu(\Theta; \beta)) \right\} \right\}.
\]

The proof of theorem 1 relies on the solution of the following variational problem: let \(H([\Theta_1; \Theta_2])\) be the real Hilbert space of all absolutely continuous functions on \([\Theta_1; \Theta_2]\) with square integrable derivative \(z' \in L^2([\Theta_1; \Theta_2])\) and inner product \((y, z) = (y, z)_2 + + (y', z')_2\); denoting by \(\|z\|_2 = (z, z)_2^{1/2}\) the \(L^2\)-norm of \(z\), we then define on \(H([\Theta_1; \Theta_2])\) the two functionals

\[
2 V(z) = \gamma z(\Theta_1) + \delta z(\Theta_2) + \|z'|_2^2 + 2\beta \int_{\Theta_1}^{\Theta_2} d\Theta A(\Theta) G(z(\Theta)),
\]

\[
C(z) = \|z\|_2^2 - \mu.
\]

Let \(\mathcal{C}_\mu([\Theta_1; \Theta_2])\) be the subset of \(H([\Theta_1; \Theta_2])\) consisting of all functions such that \(C(z) = 0\). We then have the following:

Theorem 2. Under the same assumptions as in theorem 1, the variational problem

\[
\inf_{z \in \mathcal{C}_\mu([\Theta_1; \Theta_2])} V(z)
\]

possesses a solution \(z^{*}_\mu\).