Exact Bound States for the Central Fraction Power Singular Potential $V(r) = \alpha r^{2/3} + \beta r^{-2/3} + \gamma r^{-4/3}$.

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Summary. — We obtain here a set of exact bound-state solutions, out of infinite exact bound-state solutions, for the central fraction power singular potential $V(r) = \alpha r^{2/3} + \beta r^{-2/3} + \gamma r^{-4/3}$ by using a suitable ansatz. The bound-state solutions obtained here are normalizable and for each solution there is an interrelation between the parameters $\alpha, \beta, \gamma$ of the potential and the orbital angular-momentum quantum number $l$.

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Considerable effort has been made in recent years towards obtaining the exact solution of the Schrödinger equation for certain potentials of physical interest [1-6]. These exact solutions can serve as benchmark for testing the accuracy of other non-perturbative methods. The present work is devoted to the central fraction power singular potential

$$V(r) = \alpha r^{2/3} + \beta r^{-2/3} + \gamma r^{-4/3}.$$  

The fraction power potentials have been successfully used in particle physics phenomenology. The potential (1) may also be useful in other physical problems. We shall obtain here some exact bound-state solutions, out of the infinite possible bound states, for this potential.

Consider the radial part of the Schrödinger equation ($\hbar = 2m = 1$)

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - \frac{l(l+1)}{r^2} R + \left\{ E - (\alpha r^{2/3} + \beta r^{-2/3} + \gamma r^{-4/3}) \right\} R = 0,$$

with the central potential (1). The solve the differential equation (2), we use the ansatz

$$R(r) = \exp \left[ \frac{3}{4} \alpha r^{4/3} + \frac{3}{2} \beta r^{2/3} \right] \sum_{n=0} \sum_{\nu} a_n r^{2n/3 - \nu},$$

where $a, b$ and $\nu$ are constants to be chosen suitably. On substitution in relation (2),
we see that the expansion coefficients, \( a_n \), must satisfy the three-term recurrence relation

\[ A_n^* a_n + B_n^* a_{n+1} + C_n^* a_{n+2} = 0 , \]

where

\[ A_n^* = \left( \frac{4n}{3} + 2\nu + \frac{7}{3} \right) a + b^2 - \beta , \]

\[ B_n^* = \left( \frac{4n}{3} + 2\nu + \frac{5}{3} \right) b - \gamma , \]

\[ C_n^* = \left( \frac{2n}{3} + \nu \right) \left( \frac{2n}{3} + \nu + 1 \right) - \ell(l + 1) , \]

and we have set

\[ a^2 = \alpha \quad \text{or} \quad a = \pm \sqrt{\alpha} \]

and

\[ 2ab = -E . \]

We see that \( R(r) \) is finite for \( 0 < r < \infty \) if

\[ a = -\sqrt{\alpha} \]

and

\[ b = \frac{E}{2\sqrt{\alpha}} , \]

which follows from the relation (7).

Now as \( a_0 \) is the first non-vanishing coefficient in the series (3), the relation (4) gives us

\[ C_0^* = 0 \quad \text{or} \quad \nu = l, -(l + 1) . \]

For \( R(r) \) to be finite at \( r = 0 \), we have to set

\[ \nu = l . \]

Further, if the series (3) is restricted to have finite number of terms, i.e. \( a_p \neq 0 \) but \( a_{p+1} = a_{p+2} = \ldots = 0 \), the relation (4) gives us \( A_p^* = 0 \), or the energy eigenvalue \( E_p^1 \) as

\[ E_p^1 = \pm \sqrt{4\alpha} \left( \left( \frac{4p}{3} + 2l + \frac{7}{3} \right) \sqrt{\alpha + \beta} \right)^{1/2} , \]

where we have used the relations (5a), (8), (9) and (11).

Again, for a non-trivial solution of the recurrence relation (4), \( A_n^* \), \( B_n^* \) and \( C_n^* \)