On the Unphysical Region in Dispersion Relations (*).

R. F. STREATER

Mathematics Department, Imperial College of Science and Technology - London

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Summary. — Conditions on the masses of four particles are written down such that present methods can prove inelastic dispersion relations for processes of the type \(A + B \rightarrow C + D\). One necessary condition is that both elastic relations \(A + B \rightarrow A' + B'\) and \(C + D \rightarrow C' + D'\) can be proved for forward scattering at least. The masses of \(p n \pi \pi^0\) satisfy the conditions, and also photomeson production, and the model \(K\)-meson system \(K = 3\Pi/2, A \sim N\) (where these are the masses of the particles). Using dispersion relations and the unitarity condition for the unphysical region of this model system, integral equations for the unphysical region in \(K N \rightarrow A\pi\) relations are obtained; these equations have solutions in terms of physical \(S\) matrix elements. It is also possible to set up a model \(K N \rightarrow K N\) system such that the usual dispersion relations can be proved by present methods, and also all the inelastic relations which the method of this paper needs in order to interpret the unphysical region. However, a model can be constructed such that dispersion relations can be proved, but such that their unphysical region cannot be interpreted by present methods.

1. — Introduction.

A difficulty which occurs in the theory of dispersion relations is the interpretation of the unphysical region in terms of physical data. When a relation is rigorously proved, the meaning of the unphysical region is defined in the course of the proof; it will not necessarily be in terms of physical \(S\)-matrix elements: for instance, it could be given by the analytic continuation of a

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matrix-element of a product of field operator between specified states. This mathematical definition must be related to experimentally determinable quantities, such as scattering amplitudes. H. LEHMANN has shown (1) that for π-n elastic scattering the unphysical region occurring in the non-forward dispersion relation is uniquely determined by the nearly-forward amplitudes at the same energy: it is in fact determined by analytic continuation in the momentum transfer. LEHMANN has shown (1) that this continuation is valid just in the region where the dispersion relations are proved, which is a satisfactory result. He proves that

\[ \text{Im } T(W, \theta) = \int_{-1}^{+1} d(\cos \alpha) \int_{y - \cos(\theta - \alpha)}^{\infty} dy \frac{\Phi(W, \cos \alpha, y)}{y - \cos(\theta - \alpha)}, \]

where \( W \) is the energy, and \( \theta \) the scattering angle; regularity in \( \cos \theta \) follows, as \( \theta \) enters only in the kernel \( 1/(y - \cos(\theta - \alpha)) \). Note that \( \text{Im } T \) is not at all regular in \( W \), since \( \Phi \) is an arbitrary distribution. This is illustrated in Fig. 1. Here \( p_1, k_1 \), are the initial momenta of the nucleon and meson, and \( p_2, k_2 \) the final momenta. Spin and isotopic spin are inessential complications, and can be ignored provided all the selection rules are obeyed. We use the time-like metric \( (+1 - 1 - 1 - 1) \); the scattering is determined by the two invariants \( W^2 = (p_1 + k_1)^2 \) and \( \Delta^2 = -\frac{1}{4}(p_1 - p_2)^2 \), the « energy » and « momentum transfer ». In Fig. 1, \( DAEF \) is the physical region; dispersion relations are valid if \( \Delta < \Delta_{ \text{max} } \). \( DABC \) is the region occurring in the dispersion relations, of which \( ABE \) is unphysical. A point \( X \) of \( ABE \) is reached by continuing the physical region \( YZ \) in \( A, W \) being fixed.

Although K-meson dispersion relations have not yet been proved, it is worth considering the same problem here. For the sake of argument we deal with a model K-meson of mass \( 3\Pi/2 \) (\( \Pi \) is the mass of the \( \pi \)-meson); then dispersion relations for KN-KN can be proved, for example, by the method

\[ \text{(1) H. LEHMANN: Nuovo Cimento, 10, 597 (1958).} \]