Conservation Laws in General Relativity.

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1. - Introductory considerations.

In this short note we shall show, by means of a suitable method of projection and by adopting the relative standard quantities defined in two previous papers \((1)\), that conservation principles can be formulated for any possible physical system in a gravitational field. This formulation does not yet give conservation laws without sources, like those of Einstein and many other authors (I shall here simply mention the important result recently obtained by C. Moller \((2)\)). However the present formulation, by its simplicity and generality and its purely tensor character, proves the physical content of the relative standard quantities and shows the advantages of the «projection method».

Let \(V_4\) be the space-time manifold, \(x^i\) \((2)\) physically admissible co-ordinates; \(ds^2 = g_{ik} dx^i dx^k\) the space-time metric \((+++\ldots)\); \(\gamma\) the unitary vector oriented like the line \(x^4\) \((\gamma^0 = 0, \gamma^4 = 1/\sqrt{g_{44}}, \gamma_i = g_{i4}/\sqrt{g_{44}})\); \(T_x\), the tangent vector space at the point \(x\), \(\Theta\) and \(\Sigma\) \((\text{or sometime simply } \Theta \text{ and } \Sigma)\) the sub-spaces of \(T_x\) respectively parallel and orthogonal to \(\gamma\). The \(\infty^0\) ideal particles having the lines \(x^4\) as world lines form the physical frame of reference \(S\) associated to the co-ordinates \(x^i\).

Every vector \(V\) of \(T_x\) can be uniquely decomposed into two vectors, \(V = A + N\), belonging respectively to \(\Theta\) and \(\Sigma\):

\[
A_i = \gamma_{ik} V^k, \quad N_i = \gamma_{ik} V^k,
\]

where the tensors \(-\gamma_{ik}\) and \(\gamma_{ik} = g_{ik} + \gamma_{ik}\) act as time-projector and space-projector respectively. Similarly, since the symbolic decomposition \(T_x = \Sigma + \Theta\) induces on \(T_x \otimes T_x\), the decomposition \(T_x \otimes T_x = \Sigma \otimes \Sigma + \Sigma \otimes \Theta + \Theta \otimes \Sigma + \Theta \otimes \Theta\), every double tensor \(A_{ij}\) can be uniquely decomposed into the sum of four tensors belonging respectively to the above mentioned sub-spaces of \(T_x \otimes T_x\). These tensors will be

\((1)\) C. Cattaneo: *Nuovo Cimento*, 10, 313 (1958) and 11, 733 (1959). These papers will be in the following quoted as I and II.


\((3)\) Latin indexes vary from 1 to 4; greek indexes from 1 to 3.
called the four natural projections of \( A_{ij} \) and they will be indicated by \( \mathcal{P}_{\Sigma\Sigma}(A_{ij}) \), \( \mathcal{P}_{\Theta\Theta}(A_{ij}) \), \( \mathcal{P}_{\Theta\Sigma}(A_{ij}) \), \( \mathcal{P}_{\Sigma\Theta}(A_{ij}) \). They can be obtained from \( A_{ij} \) by means of the two projectors \(-\gamma_{ij}\) and \(\gamma_{ik}\) in the following way:

\[
\begin{align*}
\mathcal{P}_{\Sigma\Sigma}(A_{ij}) &= \gamma_{ir}\gamma_{js}A^{rs}, \\
\mathcal{P}_{\Theta\Theta}(A_{ij}) &= -\gamma_{ir}\gamma_{js}A^{rs}, \\
\mathcal{P}_{\Theta\Sigma}(A_{ij}) &= -\gamma_{ir}\gamma_{js}A^{rs}, \\
\mathcal{P}_{\Sigma\Theta}(A_{ij}) &= \gamma_{ir}\gamma_{js}A^{rs}.
\end{align*}
\]  

For a symmetrical tensor \( A_{ij} = A_{ji} \) the natural decomposition may be written

\[
A_{ij} = \mathcal{A}_{ij} + \tilde{\mathcal{A}}_{ij} + \gamma_{ij}\mathcal{A}_{jj} + A\gamma_{ij},
\]

where we have put \( \mathcal{A}_{ij} = \mathcal{P}_{\Sigma\Sigma}(A_{ij}) = \gamma_{ir}\gamma_{js}A^{rs}, \tilde{\mathcal{A}}_{ij} = -\gamma_{ir}\gamma_{js}A^{rs}, \mathcal{A} = \gamma_{rs}A^{rs} \) (the symbol \( \sim \) indicates the spatial character of the tensors and vectors). Similar considerations can be applied to tensors of any order.

Some important properties of the physical frame of reference \( S \) can be characterized by means of the following tensors: the space-time vortex tensor, \( \Omega_{ij} = \partial_i\gamma_j - \partial_j\gamma_i \); the space vortex tensor \( \Omega_{ij} = \mathcal{P}_{\Sigma\Sigma}(\Omega_{ij}) = \gamma_{ir}\tilde{\partial}_r\gamma_{js} - \tilde{\partial}_r\gamma_{js}\gamma_{ir} \); \( (\tilde{\partial}_r\gamma_{ij} = 0) \); the curvature vector of the lines \( x^4 \), \( \tilde{\partial}_r\gamma_{ij} = \gamma_{ir}\gamma_{js} - \gamma_{js}\gamma_{ir} \); the tensor of Killing, \( K_{ij} = \nabla_i\gamma_j + \nabla_j\gamma_i \); the spatial tensor of deformation or Born tensor \( \mathcal{K}_{ij} = \mathcal{P}_{\Sigma\Sigma}(K_{ij}) = \gamma_{ir}\tilde{\partial}_r\gamma_{js} \); \( (\tilde{\partial}_r\gamma_{ij} = 0) \). For instance the vanishing of \( \Omega_{ij} \), or of \( \mathcal{K}_{ij} \), or of \( \tilde{\Omega}_{ij} \) characterizes respectively the geodesic frames, the rigid (or stationary) frames in the sense of Born, and the spatially irrotational frames (orthogonality of the congruence of the lines \( x^4 \)).

With the same tensors we can express, as we have shown in I and II, the gravitational force \( mG_\alpha \) acting on a freely gravitating test particle:

\[
mG_\alpha = mG_\alpha' + mG_\alpha'' = mc^2\Omega_{xx}\gamma^x + mc\tilde{\Omega}_{x0}\gamma^0,
\]

\((m = m_0(1 - v^2/c^2)^{-\frac{1}{2}}, \quad v^x = dx^x/dT, \quad v^2 = \gamma_{\alpha\beta}v^\alpha v^\beta, \quad dT = -(1/c)v^x dx^x\) being the relative standard quantities \(^1\) of the particle (mass, velocity, magnitude of the velocity, time) with respect to the physical frame of reference \( S \). By means of these tensors we can also express the energy absorbed by the particle from the field per unit of relative standard time \( T \):

\[
mc^2\Omega_{xx}\gamma^x v^x - \frac{1}{2} mc\gamma^x\partial_\gamma\gamma^0 v^0 v^\beta.
\]

The two terms can be interpreted respectively as the work of the gravitational force per unit time \( T \) and as the energy directly absorbed from the frame, if it is not stationary.

2. – Conservation equations (with sources) for any physical system.

In accordance with the general definition of the relative standard interval of time between to infinitely close events \( (dT = -(1/c)\gamma_4 dx^4) \), we shall call local standard time interval, relative to a given place point \( (x^4) \) of the system of reference \( S \), the