Study of a Singular Integral Equation of Calogero.

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Summary. — It is shown that Calogero's integral equation, involving the derivative of a Cauchy principal-value integral, has two independent solutions. Contour integral representations for these solutions are given and they are shown to satisfy a second-order differential equation of a novel type.

1. Introduction.

In this paper, we propose to investigate the linear, singular integral equation

\[ \mu \varphi(x) = f(x) - \frac{d}{dx} \frac{P}{\pi} \int_{-1}^{1} \frac{dy}{y-x} a(y) \varphi(y), \]

where

\[ a(y) = (1-y^2)^{1/2} \left(1 + \frac{\alpha}{1-y} + \frac{\beta}{1+y}\right). \]

Here \( P \) means a Cauchy principal value, \( \alpha \) and \( \beta \) are real nonnegative numbers and \( f \) is a known function. The corresponding homogeneous equation \( (f = 0) \) was introduced by Calogero \(^1\) as a natural generalization of the case \( \alpha = 0 = \beta \), which had arisen in a study \(^2\) of the asymptotic density of the zeros of the Hermite polynomials.

Let us abbreviate the Calogero equation (1.1) as

\[\mu \varphi = f + C \varphi.\] (1.3)

We show in sect. 2 that, if \(f \in L^p\), \(p > 1\), then the most general solution \(\varphi\) such that \(a \varphi \in L^q\), \(1 < q < p\), is also a solution of

\[\varphi = n - Ff + \mu F \varphi.\] (1.4)

Here \(n\) is a function that is annihilated by the Calogero operator, i.e.

\[Cn = 0.\] (1.5)

We construct the most general \(n\) explicitly and show that it depends linearly on two arbitrary parameters, i.e. it is a two-dimensional manifold. In (1.4), \(F\) is a certain \(L^2\) kernel (i.e. a Fredholm kernel) that we also construct explicitly. It is a resolvent of \(C\). The interesting thing is that \(C\) is so singular that the resolvent is a very well-behaved Fredholm kernel.

Consider first the case that \(\alpha\) and \(\beta\) (in (1.2)) are strictly positive. Then it turns out that the null functions satisfying (1.5) are bounded and continuous. Moreover, it can be shown from the properties of \(F\) that, if \(f \in L^p\), \(p > 1\), \(Ff\) is also a bounded, continuous function. Hence we may apply the classical Fredholm analysis (i.e. in \(L^2\)) to (1.4) and then, because of the properties of \(F\), conclude that the \(L^2\) solution is actually bounded and continuous. The characteristic values of \(F\) constitute a discrete set and, even when \(\mu\) is in this set, we may find a two-parameter family of solutions of (1.4), for \(n\) can be chosen such that \(n - Ff\) is orthogonal to the adjoint eigenfunction: this leaves one degree of freedom and another is obtained by adding an arbitrary multiple of the homogeneous solution itself.

When one or both of the parameters, \(\alpha\) and \(\beta\), are zero, the equation is more singular, but the general results are very similar. In particular, one may always find two independent solutions, for any value of \(\mu\). Details are given in sect. 2.

In sect. 3, we write down contour integral representations of solutions of the homogeneous Calogero equation. The expression is essentially that of Calogero (1), the only subtlety being the choices of contour. In sect. 4 we show, firstly from the integral representation of a generic solution and secondly directly from the homogeneous integral equation, that the solutions satisfy a linear homogeneous differential equation of a novel kind: beside the function \(\varphi(x)\) and its first two derivatives, it involves \(\varphi(\pm 1)\) and \(\varphi'(\pm 1)\). Finally, in sect. 5, we use a Fourier sine transform to derive a generating function, which satisfies a first-order linear differential equation and from which a solution of the Calogero equation may be obtained.