Vector and Tensor Representations of the Poincaré Group.

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Summary. — The representations of the Poincaré group on the space of vectors $A_{\mu}(K)$ and symmetric tensors $h_{\mu\nu}(K)$ are analysed. It is proved that for vanishing mass one cannot have a unitary vector (tensor) representation with helicity $\pm 1$ ($\pm 2$). To reconcile covariance and unitarity the concept of induced representations is introduced. It is shown that one can get induced unitary representations of helicity $\pm 1$ and $\pm 2$ from the vector and tensor representations, which are essentially those used in Gupta's quantization of Maxwell and Einstein equations. Thus Gupta method is seen to be intimately connected with the requirement of unitarity of the representation of the Poincaré group. Therefore its introduction can be made independent of any condition of the Hilbert-Lorentz type, since in a group-theoretical framework it follows in a natural way.

1. Introduction.

Poincaré invariance is one of the basic assumptions of quantum field theory. The most direct way of imposing it consists of identifying the physical states with rays of a Hilbert space in which a unitary representation of the Poincaré group is defined. As long as massive particles are involved, this can be accomplished in a covariant way (i.e. assuming a space of tensors as the Hilbert space of states) without any difficulty.

The situation is rather different for quantum electrodynamics and the quantized theory of gravitation, which involve massless particles of helicity $\pm 1$ and $\pm 2$, respectively. The most suitable Hilbert spaces for a covariant description of these particles should be, respectively, the space of vectors and...
the space of tensors

\[ H_1 = \{ A_\mu(K) ; A_\mu \text{ is square integrable on the light-cone} \}, \]

\[ H_2 = \{ h_{\mu\nu}(K) ; h_{\mu\nu} \text{ is square integrable on the light-cone} \}, \]

which transform in the following way under the Poincaré group:

\begin{align*}
(1) & \quad U(a, A)A_\mu(K) = \exp [iKa] A_\mu A_\nu (A^{-1}K), \\
(2) & \quad U(a, A)h_{\mu\nu}(K) = \exp [iKa] A_\mu A_\nu h_{\rho\sigma} (A^{-1}K). 
\end{align*}

Equations (1) and (2) define a representation of the Poincaré group on \( H_1 \) and \( H_2 \). These representations, however, turn out not to be unitary, and the reason for this can be seen to lie in the fact that photons and gravitons have zero mass.

More precisely, the bilinear invariant forms that one can define in \( H_1 \) and \( H_2 \) are positive definite only on subspaces of \( H_1 \) and \( H_2 \) which do not correspond to helicity ±1 and ±2, respectively. Then, since it seems desirable to have unitarity while preserving eqs. (1) and (2), we investigate which unitary representations may be induced in a Hilbert space \( H \) by the representations defined in eqs. (1) and (2). The meaning of the word "induced" will be made precise below.

Thus we find all the possible unitary representations which can be induced by the standard representations (1) and (2). Two of them are well known in the literature since they correspond to Gupta’s approach to quantum electrodynamics (1-2) and the quantized theory of gravitation (3-4). We observe that Gupta’s method is usually introduced starting from the Hilbert-Lorentz conditions imposed on the quantized fields \( A_\mu(x) \) and \( h_{\mu\nu}(x) \) (1-3). The physical states are selected by requiring that these conditions be satisfied for the positive-frequency part. Here we show that this procedure can be replaced by a more direct one, and that the identification of the physical states coincides with the search for the induced unitary representations with helicity ±1, ±2, respectively. In other words, Gupta’s method follows in a natural way from the requirement of unitarity of the representation of the Poincaré group.

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