Local Order and Onset of Chaos for a Family of Two-Dimensional Dissipative Mappings (*)

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Summary. — We study the stochastic transition of a family of dissipative mappings of the two-dimensional torus, having a pure rotation and an Anosov hyperbolic automorphism as limit cases. Numerical experiments show that the onset of chaos is characterized by a sudden destruction of basins of previously conserved invariant sets and by the appearance of a strange attractor. The nature of these phenomena is clarified by analytical considerations.

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1. - Introduction.

We study a one-parameter family \( \{ T_\theta \} \) of mapping of the two-dimensional torus \( \mathcal{T}^2 \) (represented by the unitary periodic square) given for \( 0 < \theta < \pi/2 \) by the matrices

\[
\begin{pmatrix}
2 \cos \theta & \cos \theta - \sin \theta \\
\cos \theta + \sin \theta & \cos \theta
\end{pmatrix}.
\]

(*) To speed up publication, the authors of this paper have agreed to not receive the proofs for correction.
More explicitly, if \( u \in \mathcal{F}^2 \) and \( u_{n+1} = T_\theta u_n \), or

\[
(2') \quad u_{n+1} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = T_\theta \begin{pmatrix} x_n \\ y_n \end{pmatrix},
\]

we have

\[
(2'') \quad \begin{cases} x_{n+1} = (2 \cos \theta x_n + (\cos \theta - \sin \theta)y_n) \mod 1, \\ y_{n+1} = ((\cos \theta + \sin \theta)x_n + \cos \theta y_n) \mod 1. \end{cases}
\]

In the following, provided that no confusion may arise, we shall use \( T_\theta \) for both the mappings \( (2) \) and the matrices \( (1) \).

Since \( \det |T_\theta| = 1 \) for every \( \theta \), the \( T_\theta \) preserve the Lebesgue measure \( d\mu = dx\,dy \) in the plane \( \mathbb{R}^2 \), while, due to the mod 1 operation, \( d\mu \) is preserved on the torus only for matrices with integer elements, i.e. in the extreme cases \( T_\theta \) and \( T_{\pi/2} \). For \( 0 < \theta < \pi/2 \) the mappings are, therefore, contracting. In other words, we are studying a family of dissipative dynamical systems with conservative limits.

The two extreme cases are well known: for \( \theta = 0 \), we have the classical hyperbolic automorphism of Anosov

\[
T_0 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},
\]

a dynamical system with optimal ergodic properties (mixing, completely positive entropy, exponential decay of correlations, etc.). On the other side, for \( \theta = \pi/2 \), we obtain

\[
T_{\pi/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]

i.e. a pure rotation: every point in \( \mathcal{F}^2 \) belongs to a periodic orbit of period 4, and the whole torus is continuously foliated into invariant sets, the orbits themselves. This map may be seen, therefore, as a discrete analogous of a completely integrable Hamiltonian system \(^1\).

The opposite features of \( T_\theta \) and \( T_{\pi/2} \) and the continuous dependence on \( \theta \) ensure that a stochastic transition must take place for intermediate values of the parameter. At a first stage, we shall study this transition by numerical methods, proving that it coincides with the passage of the system from elliptic