Note on Bäcklund Transformations, Dirac Factorization and the Sine-Gordon Equation (*)).

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**Summary.** — Bäcklund-type transformations for the sine-Gordon equation in $3+1$ dimensions are systematically found by exploiting various Dirac-type factorings of the d'Alembertian.

One of the most frequently occurring nonlinear equations in physics is the so-called sine-Gordon equation (SG) arising in the study of nonlinear optics, superconductivity, elementary-particle models and other diverse branches of physics (1). Its $(1+1)$-dimensional version has figured prominently in the emergence of the theory of solitons (2,3) and the development of essentially nonperturbative methods of solution (3,4) such as the Bäcklund transformation (5,6). Recently there has been extensive study of the sine-Gordon equa-

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tion in the more realistic higher-dimensional cases (7), culminating in the for-
mulation of its Bäcklund transformation equations in 2 + 1 and 3 + 1 dimen-
sions (8). The actual use of these Bäcklund transformations in obtaining solu-
tions is somewhat hindered, however, by their mathematical complexity. The
purpose of this brief note is to suggest a more physically motivated derivation
resulting in a simpler form for the Bäcklund transformation of the sine-Gordon
equation in 3 + 1 dimensions (9),

\begin{equation}
\frac{\partial^2}{\partial t^2} \psi(x) - \nabla^2 \psi(x) + \sin \psi(x) = 0.
\end{equation}

The utility of the Bäcklund transformation is that it replaces the problem
of solving a second-order nonlinear equation with that of solving a set of
first-order equations. This is accomplished, however, at the expense of increasing
the number of unknown field variables. In this respect the Bäcklund trans-
formation is analogous to Dirac's classic factoring of the Klein-Gordon equa-
tion (10). If one takes the point of view that the Bäcklund transformation is
a type of Dirac factorization of the sine-Gordon equation, its derivation is
found to follow in a systematic manner.

Using the method due to Rund (11) of deriving Bäcklund transformations
from the equations of motion and the factoring

\[ \frac{\partial^2}{\partial t^2} - \nabla^2 = \left( \frac{\partial}{\partial t} - \mathbf{\sigma} \cdot \nabla \right) \left( \frac{\partial}{\partial t} + \mathbf{\sigma} \cdot \nabla \right), \]

we arrive at the system

\begin{align}
(2a) \quad \left( \frac{\partial}{\partial t} + \mathbf{\sigma} \cdot \nabla \right) \left( \frac{\psi + \psi'}{2} \right) &= \mathbf{A} \sin \frac{\psi - \psi'}{2}, \\
(2b) \quad \left( \frac{\partial}{\partial t} - \mathbf{\sigma} \cdot \nabla \right) \left( \frac{\psi - \psi'}{2} \right) &= -\mathbf{A}^{-1} \sin \frac{\psi + \psi'}{2},
\end{align}


(9) We use the notation \( x = (x^\mu) = (x^0, x^r) = (t, \mathbf{x}) \) with \( c = 1 \) and metric with signa-
ture \( (1, -1, -1, -1) \). We write \( \partial_\mu \) for \( \partial/\partial x^\mu \) and use the summation convention
throughout.


(11) H. Rund: *Bäcklund Transformations, the Inverse Scattering Method, Solitons, and
Their Applications*, in *Lecture Notes in Mathematics*, No. 515, edited by R. M. Miura