Mie’s Electrodynamics without Potentials.

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Summary. -- It is shown that Mie’s theory of electromagnetism can be formulated without an explicit use of the four-vector potential functions on the lines developed by Infeld and Plebanski. A general form of the relationship between Mie’s quantitative functions and the electromagnetic intensities is suggested.

1. - Introduction.

It is well known that the second set of Maxwell’s equations:

\[ \sum_{\text{cyclic}} \hat{\epsilon} F_{ij} = 0, \]

implies the existence of a four-vector potential \( q_i \), such that

\[ F_{ij} = q_{ij,i} - q_{i,j} \]

(see Appendix), where the comma denotes partial differentiation with respect to the co-ordinates. However, one of the objections raised against G. Mie’s theory of electromagnetism (ref.’s (1,2)) is that it endows \( q_i \) with an objective physical significance (e.g. ref. (3)).

Infeld and Plebanski (ref. (4), referred to as IP below) have constructed

a theory of electrodynamics without explicit use of the potential functions, on the assumption that the density $\varrho$ of charged matter is proportional to the charge density $\varrho_\varepsilon$. They take the Lagrangian to be an arbitrary function $\Omega = \Omega(F, \varrho)$ of $\varrho$ and of the fundamental electromagnetic invariant

$$ F = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} . $$

The field equations of electrodynamics result from equating to zero the coefficients of the independent, arbitrary variations $\delta a_{\mu\nu}, \delta F^{\mu\nu}$ in the variation

$$ \delta \int \sqrt{-a} \Omega \, d\tau = 0 , $$

where $a_{\mu\nu}$ is the Riemann metric tensor, $a$ is the determinant of the matrix $\{a_{\mu\nu}\}$ and $d\tau$, the four-dimensional volume element of the space-time continuum. In the IP theory, the tensor indices are raised and lowered with the help of $a_{\mu\nu}$. Hence, there is a relation between the variations $\delta F_{\mu\nu}$ and $\delta F^{\mu\nu}$. In Mie's theory however, the tensors $F_{\mu\nu}$ and $F^{\mu\nu}$ are supposed to have distinct significance, the former being composed from the intensities $(E, B, A, \varrho_\varepsilon)$ and the latter, from the quantitative functions $(H, D, j, \varrho_\varepsilon)$.

The IP theory therefore, requires a modification in order that it may apply to Mie's conception and it is the purpose of this note to discuss this modification.

2. Electromagnetic invariants and the variational principle.

Writing $H^{\mu\nu}$ instead of $F^{\mu\nu}$, the first set of Maxwell's equations becomes

$$ j^\mu = \varrho_\varepsilon \varepsilon^\mu = H^{\mu\nu} ; = \frac{1}{\sqrt{-a}} \frac{\partial}{\partial x^\nu}(\sqrt{-a} H^{\mu\nu}) , $$

where the semicolon denotes covariant differentiation. It is clear that if the formal independence of $F_{\mu\nu}$ and $H^{\mu\nu}$ is to be preserved (Mie's theory in fact, leads to a relation between them arising from a consideration of the Lagrangian) $\Omega$ must depend on an invariant or invariants involving explicitly both tensors. The simplest of such invariants is (ref. (1))

$$ G = -\frac{1}{4} F_{\mu\nu} H^{\mu\nu} . $$

Letting $\varrho = k \varrho_\varepsilon$, where $k$ is a constant, we find (ref. (4)) that

$$ \varrho = k \{ a_{\mu\nu} H^{\mu\alpha} ; ; H^{\nu\beta} \} . $$

Assume now that the Lagrangian invariant $\Omega$ is an arbitrary function of $G$