Higher Born Approximations in Non-Relativistic Coulomb Scattering.

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Summary. — In this paper we calculate the first three Born approximations to the differential cross-section for non-relativistic scattering by a Yukawa potential in the limit of zero screening. The result agrees with the exact (Rutherford) cross-section for Coulomb scattering. This supports the suggestion made by Dalitz that the (divergent) higher Born approximations for Coulomb scattering act solely as a phase-factor multiplying the first Born approximation matrix-element.

1. — Introduction.

The exact solution of the Schrödinger equation for scattering in a Coulomb potential was first given by Gordon (1) who found the asymptotic form

$$\psi(r) \sim \left[1 - \frac{\gamma^2}{ik(r - z)}\right] \exp[ikr + i\gamma \log k(r - z)] -$$

$$- \frac{I(1 + i\gamma)}{I(1 - i\gamma)} \gamma \cosec^2 \frac{\theta}{2} \frac{1}{kr} \exp[ikr - i\gamma \log k(r - z)],$$

where $\gamma = ZZ'e^2 / 4\pi\hbar^2$, $k = mv / \hbar$ and $\theta$ is the scattering angle. This leads to the well-known Rutherford differential cross-section.

If an attempt is made to obtain this result by means of the Born approximation method, it is found that the first approximation leads to the exact result, and that the higher approximations are all infinite (as a consequence

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of the infinite forward scattering amplitude in the first Born approximation). Dalitz has suggested that if the Born approximation series for the scattering amplitude is evaluated for a screened (Yukawa) potential, then in the limit of zero screening it will reduce to the first Born approximation amplitude multiplied by an (infinite) phase factor of unit modulus, and so will lead to the exact result for the differential cross-section.

Dalitz calculated the most divergent parts of the first three approximations and showed that they were equivalent to a phase factor, as suggested. In the present paper the first three approximations are calculated exactly, in the zero-screening limit, and again we find that these lead to the exact result, to this order in \( Z \).

2. - Formulation of the problem.

We require to solve the Schrödinger equation

\[
\frac{1}{2m} (-\nabla^2 - p^2) \psi(r) = \frac{Ze^2}{4\pi} \frac{\exp[-\lambda r]}{r} \psi(r),
\]

for scattering of an electron by a nuclear charge \( Ze \). Note that we take \( \hbar = 1 \), and work with rationalized units. \( \lambda \) is the screening constant, and we work in the limit \( \lambda \to 0 \). We impose the boundary condition

\[
\psi(r) \sim \exp[i \mathbf{p} \cdot \mathbf{r}] + \frac{1}{r} g(\theta) \exp[ipr].
\]

Note that this asymptotic behaviour disagrees with that in a pure Coulomb field, as given by Gordon. This is probably the root cause of the difficulties encountered in the Born approximation method for a Coulomb potential. However for a screened potential this is the correct behaviour to impose at infinity.

We then get the integral equation

\[
\psi(r) = \exp[i \mathbf{p} \cdot \mathbf{r}] + \frac{1}{4\pi} \int d^3r' \exp[i \mathbf{p} \cdot (\mathbf{r} - \mathbf{r}')] \cdot \frac{2mZe^2}{4\pi} \exp[-\lambda r'] \cdot \frac{\psi(r')}{r'}.\]

We transform to momentum space by defining

\[
\varphi(s) = \frac{1}{(2\pi)^3} \int d^3r \exp[-i s \cdot r] \psi(r),
\]