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General Relativity without Coordinates.

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Summary. — In this paper we develop an approach to the theory of Riemannian manifolds which avoids the use of co-ordinates. Curved spaces are approximated by higher-dimensional analogs of polyhedra. Among the advantages of this procedure we may list the possibility of condensing into a simplified model the essential features of topologies like Wheeler's wormhole and a deeper geometrical insight.

1. – Polyhedra.

In this section we shall first describe our approach for the simple case of 2-dimensional manifold (surfaces). Following ALEKSANDROV [1] we develop the theory of intrinsic curvature on polyhedra. A general surface is then considered as the limit of a suitable sequence of polyhedra with an increasing number of faces. A rigorous definition of limit is not given here since it would involve a treatment of the topology on the set of all polyhedra and this would carry us too far. It is to be expected however that any surface can be arbitrarily approximated, as closely as wanted, by a suitable polyhedron. The approximation will be bad if we look at the details to the picture but an observer looking at the broad details only will find it quite satisfactory. On any surface we can define an integral Gaussian curvature by carrying out curvature experiments with geodesic triangles.

Let t be one such a triangle and let α, β, γ be its internal angles. If the geometry inside the triangle is not euclidean we have in general $\alpha+\beta+\gamma\neq\pi$.

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(1) P. S. ALEKSANDROV: *Topologia combinatoria* (Torino, 1957).

The Gaussian integral curvature ε_t of t is then defined by $\varepsilon_t = \alpha + \beta + \gamma - \pi$. On a sphere we have $\varepsilon_t = A_t/R^2$, where R is the radius of the sphere and A_t the area of t. If t shrinks to a point P and the limit $\lim\limits_{A\to 0} \varepsilon_t/A_t = K(P)$ exists and is independent of the particular limiting procedure we take $K(P)$ as definition of local Gaussian curvature in P. On a sphere obviously $K(P) = 1/R^2$. It is well known that $\varepsilon_t = \int\limits_t K(P)\,\mathrm{d}A$, where $\mathrm{d}A$ is the area element on the surface and the integration is carried inside t. Let it be clear however that, while the integral curvature exists under very broad conditions, the local curvature may not exist at all as an ordinary function but rather only as a measure.

This last point is essential in defining the curvature of a polyhedron M. It is obvious that if t lies entirely on a face of M we have $\varepsilon_t = 0$. The same result holds if t does not contain any vertex of M. Therefore $K(P) = 0$ if P is not a vertex.

If t contains a vertex, say V, and no other vertices, it will not depend on the explicit form of t but only on the particular choice of V. In other words $\varepsilon_t = \varepsilon_V$ is a characteristic constant of V (the deficency of V) ε_V can be found as follows. Take the sum of all internal angles of the faces of M with vertex V. This sum equals $2\pi - \varepsilon_V$. If several vertices V, X, etc., are inside t we have $\varepsilon_t = \varepsilon_V + \varepsilon_X + \varepsilon_Y + \text{etc.}$

All these results can be condensed into the original formula $\varepsilon_t = \int\limits_t K(P)\,\mathrm{d}A$, provided one understands $K(P)$ as a Dirac type distribution, having the vertices as supports. The integral $\int\limits_M f(P)K(P)\,\mathrm{d}A$, where $f(P)$ is a continuous function, is then to be calculated as $\sum\limits_n f(V_n)\varepsilon_n$, where ε_n is the deficency of V_n, and the summation is carried out on all vertices of M.

If M is a compact (*i.e.* finite closed) polyhedron Gauss-Bonnet's integral curvature theorem can be written as

$$\sum_n \varepsilon_n = 2\pi(2 - N)\,,$$

N is here the genus of M, $N = 0$ for a sphere and $N = 2$ for a torus. There is no loss in generality in supposing that all faces of M are triangles.

Under this form the connection of this theorem with Euler's formula for the genus is most evident. Indeed let σ_{fn} be the internal angle of the face f with the vertex n. We have:

$$\sum_f \sigma_{fn} = 2\pi - \varepsilon_n\,,$$

where the summation is carried out on all faces f having V_n as vertex. It is