SERVICE NETWORKS WITH MULTICHANNEL NODES IN A RANDOM ENVIRONMENT

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Service networks with multichannel nodes of semi-Markovian type are considered. The parameters of a source of demands depend on the state of the Markovian random environment. For the process of servicing demands, the conditions of existence of a stationary mode are found, and the properties of stationary distribution in terms of spectral characteristics of the routing matrix are investigated.

Keywords: service networks, analysis of queues, Markov chains, Markovian random environments, service networks with multichannel nodes.

Let us consider a service network consisting of r nodes. One common input flow of demands, which is controlled by a Markov chain \( \eta(t) \in [1, 2, ..., N] \), arrives at service nodes from outside according to the algorithm described below.

The moments of arrival of demands coincide with the moments \( t_n, n = 1, 2, ..., \) of changing states of the chain \( \eta(t) \). If the chain \( \eta(t) \) passes to a state \( i \) at a moment \( t_n \), then the demand with a number \( n \) is serviced in a node \( j \) with probability \( h_{ij} \); note that \( \sum_{j=1}^{r} h_{ij} = 1 \) and \( H = \| h_{ij} \| \) is a rectangular \( N \times r \) matrix.

There is an unlimited number of servers of the same type at each node, and \( G_j(t), j = 1, 2, ..., r \), is the distribution function of the service time at the \( j \)th node. After servicing at the \( j \)th node, the request passes to a node \( k \) with probability \( p_{jk} \) and quits the network with probability \( 1 - \sum_{k=1}^{r} p_{jk} \); here, \( P = \| p_{jk} \| r \) is the routing matrix of the network.

According to a classification of networks, which is used in [1], the model described above is a Jackson network with nodes of the type \( M | GI | \infty \) and a source of demands of the first kind. Parameters of the source of demands depend on states of the Markovian random environment \( \eta(t) \). Networks of such a type frequently arise during the investigation of processes of processing and transferring of information in data-processing and communication networks.

The objective of this article is to study existence conditions and characteristics of the stationary distribution for the process of demand service by the stochastic network described above.

Let \( X_j(t), j = 1, 2, ..., r \), be the number of occupied devices at the \( j \)th node at an instant of time \( t \geq 0 \), \( X(t) = (X_1(t), ..., X_r(t)) \). Denote by \( X_j^i(t), i = 1, 2, ..., N, j = 1, 2, ..., r \), the number of occupied devices at the \( j \)th node at an instant of time \( t \geq 0 \) if the network is empty at the initial moment and \( \eta(0) = i \) and \( X^i(t) = (X_1^i(t), ..., X_r^i(t)) \). Denote the multidimensional generating functions of vectors \( X(t) \) and \( X^i(t) \) by \( \Phi(t, z) \) and \( \Phi^i(t, z) \), respectively, where \( z = (z_1, ..., z_r) \), \( |z| \leq 1 \). If a stationary mode of operation exists for the service network, then \( X_j, j = 1, 2, ..., r \), is the number of occupied devices at the \( j \)th node in the stationary mode of operation, and \( \Phi(z), |z| \leq 1 \), is the generating function of a vector \( X = (X_1, ..., X_r) \).

The trajectory of a demand from the moment of arrival at the network to the moment of escaping from the network can be described by a semi-Markovian process \( \xi(t), t \geq 0 \), which assumes values in a set of states \( [1, 2, ..., r, r + 1] \) and is defined by a semi-Markovian matrix.

\[ Q(t) = |Q_{ij}(t)|^{r+1}_{j=1}, \]

\[ Q_{ij}(t) = \begin{cases} G_i(t)p_{ij}, & i = 1, 2, \ldots, r; j = 1, 2, \ldots, r, r + 1; \\ \delta_{r+1}G_{r+1}(t), & j = 1, 2, \ldots, r, r + 1; \end{cases} \]

\[ G_{r+1}(t) = \begin{cases} 0, & t < 1; \\ 1, & t \geq 1, \end{cases} \]

where \( \delta_{ij} \) is the Kronecker symbol.

The state \( r + 1 \) for the process \( \xi(t) \) is absorbing. The absorption into the state \( r + 1 \) is interpreted as the escape of the demand from the network. Denote by \( p_{ij}(t) = P \{ \xi(t) = j / \xi(0) = i \} \) the transition probabilities of the semi-Markovian process \( \xi(t) \). The result given below is valid for the process of servicing demands in the network.

**THEOREM 1.** If the Markov chain \( \eta(t) \) is ergodic, the spectral radius of the routing matrix \( P \) is less than unity, \( G_i(0+) = 0, \) and \( \int_0^\infty t dG_i(t) < \infty \) for \( i = 1, 2, \ldots, r, \) then there exists a stationary mode of operation for the process \( X(t), t \geq 0, \) and, moreover,

\[ \Phi(z) = 1 - \sum_{j=1}^{N} \pi_j \prod_{j=1}^{r} \left[ p_{j,r+1}(t) + z_1 p_{j1}(t) + \ldots + z_r p_{jr}(t) \right]^j, \]

where \( \pi_1, \pi_2, \ldots, \pi_N \) is an ergodic distribution of \( \eta(t) \).

**Proof.** We assume that, at the instant of time \( t = 0, \) \( X_i(0) = k_j, \) \( i = 1, 2, \ldots, r, \) and the demands were not served before the moment \( t = 0. \) Then

\[ \Phi(t, z) = \prod_{i=1}^{N} p_i(0) \Phi_i(t, z) \prod_{j=1}^{r} \left[ p_{j,r+1}(t) + z_1 p_{j1}(t) + \ldots + z_r p_{jr}(t) \right]^j, \]

where \( p_i(0) = P \{ \eta(0) = i \} \) is the initial distribution of the Markov chain \( \eta(t). \) Since \( \lim_{t \to \infty} p_{j,r+1}(t) = 1 \) for any \( j = 1, 2, \ldots, r, \) we have

\[ \lim_{t \to \infty} \Phi(t, z) = \lim_{t \to \infty} \prod_{i=1}^{N} p_i(0)\Phi_i(t, z). \]

We note that if demands \( k_j, j = 1, 2, \ldots, r, \) at the \( j \)th node are already served before the moment \( t = 0, \) then relation (2) remains valid. The demand time at the initial state in the semi-Markovian processes connected with the service of demands \( k_j \) should obviously be changed.

The generating functions \( \Phi_i(t, z), i = 1, 2, \ldots, N, \) satisfy the system of integral equations

\[ \Phi_i(t, z) = e^{-\lambda_i t} + \sum_{j=1}^{N} \int_0^t e^{-\lambda_j u} \Delta_{ij} \Phi_j(t-u, z) \]

\[ \times \left\{ \sum_{k=1}^{r} h_{jk} \left[ p_{k,r+1}(t-u) + z_1 p_{k1}(t-u) + \ldots + z_r p_{kr}(t-u) \right] \right\} du, \]

which can be represented in the form of equations of Markovian restoration

\[ \Phi_i(t, z) = \Phi_i(t, z) + \sum_{j=1}^{N} \int_0^t \Phi_j(t-u, z) \frac{\Delta_{ij}}{\lambda_i} (1 - e^{-\lambda_i u}) du, \]

where