On the Compatibility of Relativistic Wave Equations in Riemann Spaces.

H. A. Buchdahl

Physics Department, University of Tasmania - Hobart

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Summary. — On a previous occasion it was shown that the "natural generalization" to a Riemann space $V_4$ of a certain set of flat-space free-field equations for particles of spin $S = \frac{3}{2}$ is internally consistent if and only if the $V_4$ is an Einstein space. It is now shown that, this case apart, all equations for particles of spin $S > \frac{3}{2}$ which may be said to conform to a "strong principle of equivalence" are compatible if and only if the $V_4$ is of constant Riemannian curvature. The corresponding second-order wave equations in such a space are written down. Certain modified first-order equations for the case $S = 2$ which involve the curvature tensor explicitly are shown to be consistent in an Einstein space.

1. — Introduction.

The free-field equations for particles of spin $S > 1$ have hitherto usually been considered in the context of flat space $E_4$. It is known (1) that the introduction of electromagnetic interactions by the formal prescription $\partial_k \rightarrow \partial_k - ieA_k$, where $A_k$ is the electromagnetic potential, leads to the inconsistency of the resulting field equations. It is therefore of interest to consider the introduction of gravitational interactions. In the first instance this entails the need for field equations in a form appropriate to an arbitrary Riemann space $V_4$, such that when the $V_4$ is an $E_4$ they reduce to the usual equations. The normal procedure is to replace, in the first-order flat-space equations, all ordinary derivatives by covariant derivatives. The resulting equations will be called

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the «natural generalization» of the flat-space equations; and the question arises whether they are consistent.

In an earlier paper (2) (hereafter referred to as A) it was shown that the natural generalization A(5-6) of the Pauli-Fierz equations for particles of spin \( S = \frac{3}{2} \) are in fact consistent if and only if the \( V_4 \) is an Einstein space. It now appears that just these equations possess a feature which is absent in the case of all other equations relating to spin \( S > \frac{3}{2} \), and that the result quoted above is rather specialized (*). To be explicit, the equations A(5.6) are peculiar in that neither of the field spinors which occur in them has more than two dotted or two undotted indices.

The main object of the present paper, then, is to investigate the general case; and it will be shown in Sections 2 and 3 that the equations for \( S > \frac{3}{2} \) (excepting those dealt with in A) are consistent if and only if the \( V_4 \) is a space of constant Riemannian curvature, \( S_4 \). In Section 4 the second order wave equations (in an \( S_4 \)) are then written down. In an \( E_4 \), there are \( S \) (\( S \) integral) or \( S + \frac{1}{2} \) (\( S \) half odd integral) equivalent theories for particles of spin \( S \) (3). In a \( V_4 \) one can enquire meaningfully into the validity of an analogous result only when the equations are compatible in the first place, i.e. when the \( V_4 \) is an \( S_4 \); and this is done in Section 5.

The remarks of Section 4 of A are again relevant, though the situation is now more serious in that even when the back reaction of the spinor field upon the gravitational field is disregarded the field equations will generally be incompatible since physical space is not usually of constant Riemannian curvature. It might of course be objected that the present investigation relates only to the natural generalization of the flat-space equations, and that a correct generalization might introduce such additional terms as would ensure compatibility. These additional terms would of necessity have to involve the curvature tensor (and perhaps its derived concomitants). Now, appropriately to the present context, the «strong principle of equivalence» might be stated as follows: «At any chosen point the first-order field equations reduce in natural co-ordinates to the flat-space equations»; though whether one accepts this principle at this stage is to some extent a matter of taste. If one does so then one immediately excludes thereby the existence of all particles of spin \( S > \frac{3}{2} \); or if one takes the reaction of the field on the curvature of space into account, of all particles of spin \( S > 1 \). (From the point of view of the present-day phenomenology of elementary particles this result would not

(2) H. A. Buchdahl: Nuovo Cimento, 10, 96 (1958). The formalism here is the same as that used in A.

(*) Except where the contrary is stated it will be taken as understood that all field equations considered below are the natural generalization of the corresponding equations relating to an \( E_4 \).

(3) H. Umezawa: Quantum Field Theory, Ch. iv (Amsterdam, 1956), p. 66.