Green's Function for Simple Brownian Kinetics
and Its Application to Plasma (*).

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Summary. - An analysis is outlined concerning the derivation and application of the Green's function for a class of Fokker-Planck equations with strictly time-dependent external fields and constant viscosity-diffusion coefficients, the kinetic equation commonly used in the theory of Brownian motion. The form of Green's function for this general class of partial differential equations is given explicit expression in terms of the collisionless time-development operator on the set of initial velocities under the influence of external fields and viscous force. Application of such a singular solution is made to simplify the formulation for the near-equilibrium kinetics of excitation of neutral atoms by electrons and to electromagnetic propagation problems where a linearized self-consistent field treatment of the electronic constituent of a three-fluid plasma is necessary.

1. - Green's function for simple Brownian kinetics.

The present investigation concerns the derivation of the Green's function for a class of Fokker-Planck equations with strictly time-dependent external fields and constant viscosity-diffusion coefficients. It is this type of kinetic equation which is used in the theory of Brownian motion. The utility of such a Green's function lies in the possibility of describing a variety of situations in kinetics as well-posed initial and/or boundary value problems.

The class of Fokker-Planck equations considered is typified by the form

\[
\left\{ \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \frac{F}{m} \frac{\partial}{\partial \mathbf{r}} - \frac{\partial}{\partial \mathbf{v}} \left( \beta \mathbf{v} + q \frac{\partial}{\partial \mathbf{v}} \right) \right\} f(\mathbf{r}, \mathbf{v}, t) = \varphi(\mathbf{r}, \mathbf{v}, t),
\]

(*) This work has been completed while the Author was employed at Raytheon Research Division, Waltham, Mass.
where \( \beta \) and \( q \) are respectively \(^{(1)}\) constant viscosity and diffusion coefficients related to each other by

\[
q = \beta \frac{kT}{m}.
\]

The external force \( F \) consists of strictly time-dependent (or constant) fields, magnetic \( H \), in the \( z \)-direction and electric \( E \):

\[
F = e(v/c \times H + E).
\]

The constitutive derivative of (1.1) expresses the total-time derivative of the distribution function along the collisionless trajectories determined by the equations of motion

\[
\dot{v} = F/m, \quad \dot{r} = v.
\]

Let \( \frac{D}{Dt} \) symbolize this derivative, to give (1.1) in simpler form

\[
\left\{ \frac{D}{Dt} - \frac{\partial}{\partial v} \left( \beta v + q \frac{\partial}{\partial v} \right) \right\} f = \varphi.
\]

Note that the arguments \( v \) and \( t \) are no longer Cartesian since trajectory (1.4) is implied by \( \frac{D}{Dt} \).

Upon multiplication of (4) by a two-sided function \( G(r', v', t' | r, v, t) \) and integration over the domains \( r, v, t \), the adjoint equation is derived.

It is seen that the formal adjoint equation which the Green's function must satisfy is

\[
\left( -\frac{D}{Dt} + \beta v \cdot \frac{\partial}{\partial v} - q \frac{\partial}{\partial v} \cdot \frac{\partial}{\partial v} \right) G = \delta(t - t') \delta(r - r') \delta(v - v').
\]

Recall the definition of \( \frac{D}{Dt} \) as a time derivative along a trajectory with direction numbers

\[
\dot{v} = F/m, \quad \dot{r} = v.
\]

Thus if we consider the time derivative along a related trajectory with direction numbers

\[
\dot{v} = F/m - \beta v, \quad \dot{r} = v
\]

and define this derivative as \( \mathcal{D}/Dt \) (1.6) becomes

\[
\left\{ \frac{\mathcal{D}}{Dt} + q \frac{\partial}{\partial v} \cdot \frac{\partial}{\partial v} \right\} G = -\delta(t - t') \delta(r - r') \delta(v - v').
\]

\(^{(1)}\) S. Chandrasekhar: Rev. Mod. Phys., 15, 1 (1943), Chap. II.