Remarks on Dirac Spurs and Pfaffians.

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Summary. — This paper gives a recipe based on Pfaffian techniques for writing only the nonvanishing terms in the evaluation of spurs of certain types of products of Dirac matrices.

It is well known that the task of evaluation of spurs of products of Dirac matrices encountered, e.g., in quantum electrodynamics is often laborious. There have been attempts to devise algorithms which are helpful in evaluation of spurs. To this end, CAIANIELLO and FUBINI (1) have developed elegant techniques based on Pfaffians which are closely related to and as simple as determinants. Recently CHISHOLM has obtained a useful relation in evaluation of a product of spurs of Dirac matrices (2).

In actual problems in electrodynamics, we have often to sum over pairs of $\gamma^\mu$s which bracket products of Dirac matrices. In such cases it is found that quite a number of terms cancel out each other in actual evaluation of spurs. It would be definitely advantageous to avoid from the beginning writing down such terms and this would certainly cut down the time of computation considerably. The purpose of this note is to present a method of writing down only the nonvanishing terms in such cases.

From the results of ref. (1), we know that the spur of a product of Dirac matrices reduces to a Pfaffian and so we have

\[ \frac{1}{2} \text{Sp} \left( Q_1 Q_2 \ldots Q_{2n} \right) = (12 \ldots 2n) , \]

(1)

where

\[(2)\]
\[Q_h = \gamma^\mu p^h_\mu\]
\[(\mu = 1, 2, 3, 4),\]

if \(Q_h\) is a four-dimensional vector matrix with \(p^h_\mu\) being the components of a four-vector. However, when we wish to include the mass terms also in \(Q_h\), we can define a five-dimensional vector matrix

\[(3)\]
\[Q_h = \Gamma^\mu q^h_\mu;\]

where

\[q^h_\mu = p^h_\mu;\]

and

\[(4)\]
\[q^h_5 = (-1)^h p^h_5;\]

with \(ip^h_5\) being the mass term in \(P_h\) given by

\[(5)\]
\[P_h = \gamma^\mu p^h_\mu + ip^h_5.\]

In (3) we have introduced the dual representation of \(\gamma\)-matrices:

\[(6)\]
\[
\begin{align*}
\Gamma^\mu &= i\gamma^\mu\gamma^5, \\
\Gamma^5 &= \gamma^5.
\end{align*}
\]

We note that

\[\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu = 2\delta^\mu_\nu \quad (\mu, \nu = 1, \ldots, 5).\]

Using equations from (3) to (6) we have

\[(7)\]
\[
\begin{align*}
Q_r Q_s + Q_s Q_r &= 2q^r_\mu q^s_\mu \\
&= 2(rs).
\end{align*}
\]

(7) generalizes in an obvious way the scalar product of four-vectors.

It is usual to meet spurs of products of the type

\[(8)\]
\[
\frac{1}{4} \text{Sp} (Q_1 Q_2 \ldots \gamma^\mu_1 Q_{k+1} \ldots \gamma^\mu_k Q_{l+1} \ldots Q_n),
\]

where a pair of \(\gamma^\mu\)'s bracket a product of a certain number of \(Q\)'s. Such \(\gamma^\mu\)'s represent polarization vectors of photons in quantum electrodynamics. When we sum over \(\mu\)'s it is found that a large number of terms cancel each other.