Pertinence of the Semi-Group Law in the Theory of the Decay of an Unstable Elementary Particle (*).

S. TWAREQUE ALI
International Centre for Theoretical Physics - Trieste

L. FONDA
International Centre for Theoretical Physics - Trieste
Istituto di Fisica Teorica dell'Università - Trieste

G. C. GHIRARDI
Istituto di Fisica Teorica dell'Università - Trieste
Istituto Nazionale di Fisica Nucleare - Sezione di Trieste

(ricevuto il 22 Luglio 1974)

Summary. — The problem of the decay of an unstable elementary particle is analysed from the point of view of a quantum-mechanical theory of randomly repeated measurements. A semi-group law of evolution, on the space of all density matrices, is shown to result. This situation is contrasted with some earlier work on the problem where a semi-group law of evolution on a Hilbert space of state vectors was postulated. Many of the difficulties of this older theory are naturally resolved within the context of the currently proposed theory.

I. – Introduction.

A quantum-mechanical description of the time evolution of a system of decaying unstable elementary particles in terms of a semi-group law has been an enticing idea for some time past. A mathematically rigorous exposition

(*) Supported in part by the Istituto Nazionale di Fisica Nucleare, Sezione di Trieste.
of this idea is given, for example, in ref. (13). The origin of the notion stems from the fact that the classical decay law for a system of unstable particles is given by an exponential function, i.e. the number of undecayed particles decreases exponentially with time. The most obvious quantum generalization of this law would then be to assume that the time evolution of the unstable quantum states is governed by a semi-group law.

To put things in more precise terms, let $\mathcal{H}_u$ denote the Hilbert space of vectors which represent the various possible (pure) states of a system of unstable particles. If $\psi_0^u$ is a vector in $\mathcal{H}_u$ at the time $t = 0$, then $\psi_0^u$ is assumed to evolve in time according to the law

\begin{equation}
\psi_t^u = V_t \psi_0^u,
\end{equation}

where $V_t$ is a bounded operator on $\mathcal{H}_u$ satisfying

a) $\|V_t\| < 1$ for all $t > 0$;

b) $V_0 = I$, the identity operator on $\mathcal{H}_u$;

c) $V_{t_1 + t_2} = V_{t_1} V_{t_2}$ for all $t_1, t_2 > 0$;

d) the map $t \mapsto V_t$ is continuous in the strong-operator topology.

The fact that the $V_t$'s form a contractive (condition a)) semi-group rather than a one-parameter unitary group is taken to reflect a loss of probability from the space $\mathcal{H}_u$ of unstable particles to the Hilbert space of their decay products. Let us denote this latter Hilbert space, i.e. of the decay products, by $\mathcal{H}_d$. Then $\mathcal{H}_d$ is assumed to be orthogonal to $\mathcal{H}_u$, and the full Hilbert space $\mathcal{H}$ of the problem is taken to be

\begin{equation}
\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_d.
\end{equation}

On $\mathcal{H}$ one assumes that the time evolution of the total system of unstable particles plus their decay products is unitary.

A canonical way to obtain the space $\mathcal{H}$ (in a certain minimal sense), given $\mathcal{H}_u$ and $V_t$, is to use the Sz. Nagy extension theorem (4). The construction in that theorem provides one with a projection operator $P$ and a one-parameter