The Application of the $hp$–Finite Element Method to Electromagnetic Problems

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1 INTRODUCTION

In this paper, we consider the application of finite element methods to the solution of problems in electromagnetics. The emphasis will be placed on techniques which can be readily applied in the solution of a range of practical engineering based problems, but the applications will focus on two key areas where we have previous experience, viz electromagnetic scattering and eigenvalue computation. We will draw on an extensive literature, from both the mathematical and engineering disciplines, so it is hoped that mathematicians and engineers will find this review to be both interesting and informative.

The bulk of this article will be devoted to the subject of the approximation of Maxwell’s equations by edge elements. These elements appear in the literature under a variety of different names, including vector elements and covariant projection elements, while within the mathematical community they are often referred to as $H(\text{curl})$ conforming elements. In what follows, we will mainly employ the latter term, as the edge element designation is only appropriate when talking about the lowest order case, in which each degree of freedom is associated with an element edge. For higher order elements, degrees of freedom can also be associated with the element interiors and, in three–dimensions, with the element faces so that, in this case, the edge element terminology seems inappropriate.

The mathematical and engineering literature provides extensive coverage of the use of the finite element method in electromagnetics and there are a number of text books that provide an introduction to the topic. An engineering perspective may be obtained from the text by Jin [70], which provides an introductory survey of edge elements and a description of some simple scattering and eigenvalue computations. An alternative is the work of Silvester and Ferrari [112], in which basic one dimensional Lagrangian finite elements are covered in some detail before moving on to more advanced topics, including two and three–dimensional edge elements. The book by Volakis, Chaterjee and Kempel [120] describes the basis functions for the lowest order edge elements on triangles and tetrahedra, while the applications considered include scattering and eigenvalue computations in both two and three–dimensions. Salazar–Palma, Sarkar, Garcia–Castillo Roy and Djordjevic [110] have produced a large text that considers many aspects of edge elements and includes tables of basis functions for different element types and for a selection of low polynomial degrees. Construction details for the elements also appear along with some details of error indicators and adaptive $h$–refinement strategies. Application to eigenvalue and scattering problems complements the theory.

The subject is approached from a mathematical standpoint in the book by Bossavit [31],

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which covers magnetostatics, infinite domains and eddy current problems. Here the material is presented from the perspective of differential forms. An alternative perspective is the functional approach adopted in the book by Monk [92]. This provides extensive coverage of the current mathematical theory regarding the approximation capabilities of edge based finite element methods, along with details of the construction of the elements. One chapter is completely devoted to recent algorithmic developments in the subject and engineers may find this to be particularly informative.

This article is organised as follows: In Section 2, an introductory overview of Maxwell’s equations is given. This is followed by a history of edge element approximation in Section 3. Section 4 is devoted to some of the recent developments concerning the $hp$–version of the finite element method. Dispersive properties of $H$ (curl) conforming approximations are discussed in Section 5. Section 6 describes specific applications of $H$ (curl) conforming approximations to scattering problems and Section 7 concentrates on eigenvalue computations. In Section 8, a short discussion of some alternative non–conforming finite element approaches is provided.

2 Maxwell’s Equations

Electromagnetic phenomena are governed by Maxwell’s equations. These equations relate the electric and magnetic field intensity vectors, denoted by $E^*$ and $H^*$ respectively, and the material properties of the medium. The full set of equations may be written as

$$\text{div } D^* = \gamma \quad (1)$$
$$\text{div } B^* = 0 \quad (2)$$
$$\text{curl } H^* = J^* + \frac{\partial D^*}{\partial t} \quad (3)$$
$$\text{curl } E^* = -\frac{\partial B^*}{\partial t} \quad (4)$$

To complete the set, the continuity equation

$$\text{div } J^* + \frac{\partial \gamma^*}{\partial t} = 0 \quad (5)$$

and the constitutive equations

$$D^* = \varepsilon E^* \quad B^* = \mu H^* \quad (6)$$

must be added. Here $D^*$ is the electric flux density vector, $\gamma$ is the electric charge density, $B^*$ is the magnetic flux density vector and $J^*$ is the electric current density vector. In addition, $\varepsilon, \mu, \sigma$ denote the permittivity, permeability and conductivity respectively of the medium.

In many practical problems, we can apply simplifying assumptions to the governing equation set. For present purposes, we will assume that

1. the medium obeys Ohm’s Law, so that $J^* = \sigma E^* + J^*_{\text{app}}$ where $J^*_{\text{app}}$ is the applied current density used in source type problems (we will assume that $J^*_{\text{app}}$ is zero throughout);

2. the electric charge density $\gamma = 0$;

3. the medium is non–lossy, so that $\varepsilon, \mu \in \mathbb{R}$;