Numerical solution of the fifth-order KdV equation for shallow-water solitary waves

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Summary. — A numerical solution for a solitary-wave solution of the fifth-order Korteweg-de Vries equation for shallow-water waves is found by a series solution in exponential. The solution agrees to within a few percent with the sech\(^2\) solution of the KdV equation.

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The propagation of surface waves in a shallow channel of constant depth is described to first order in two small parameters by the well-known Korteweg-de Vries equation (KdV). The KdV equation has a soliton-like solution or solitary wave of sech\(^2\) form.

The KdV equation is derived from the equations of hydrodynamics for an inviscid, irrotational, incompressible fluid. If one carries to the next order in the small parameters the derivation of the KdV equation, following the procedure of Lamb [1], one obtains an evolution equation with a fifth-order derivative, called in the literature the fifth-order KdV equation.

This equation was first obtained by Olver [2,3] and subsequently by Sachs and Pennel [4]. The equation is

\begin{equation}
\eta_t + \eta_x + \frac{3}{2} \varepsilon \eta \eta_x + \frac{1}{6} \delta^2 \eta_{xxx} - \frac{3}{8} \varepsilon^2 \eta^2 \eta_x + \frac{19}{360} \delta^4 \eta_{xxxx} +
+ \frac{23}{24} \varepsilon \delta^2 \eta_x \eta_{xx} + \frac{5}{12} \varepsilon \delta^2 \eta \eta_{xxx} = 0,
\end{equation}

where \(\eta = \eta(x, t)\) is the dimensionless surface elevation; the subscripts refer to derivatives in the corresponding variables, \(\varepsilon = a/h, \delta = h/L\), where \(a\) is of the
order of the maximum amplitude, $L$ of the order of the length of the surface disturbance, and $h$ the constant fluid depth.

Olver [5] has investigated the soliton solution of eq. (1) and shown that a series solution in polynomials in $\text{sech}^2$ does not converge formally. He has also shown the convergence of the asymptotic expansion or «solitary-wave tails» for large values of $x$. We will follow a different approach and try to find out if there is a solitary-wave solution of eq. (1), that is to say a solution of the form $\eta(x - ct)$, where $c$ is the dimensionless velocity, to be determined.

We will extend the asymptotic expansion to all values of $x$, which, for simplicity, can be evaluated at $t = 0$,

\begin{equation}
\eta(x) = \sum_{n=1}^{N} c_n \exp\left[-n\lambda x\right].
\end{equation}

The origin of the $x$ coordinate is chosen at the position of the maximum amplitude of the solitary wave. The constant $\lambda$ is determined by the characteristic equation obtained by substituting the solution $\exp\left[-x\right]$ into eq. (1) for large $x$ neglecting second-order terms,

\begin{equation}
(1 - c)\lambda^2 + \frac{1}{6}\lambda^3 + \frac{19}{360}\lambda^5 = 0.
\end{equation}

The positive root of eq. (3) gives the solution (2) for $x > 0$, the negative root gives the solution (2) for $x < 0$, hence $\eta(x)$ is a symmetric function. Substituting eq. (2) into the fifth-order (KdV) equation (1), one obtains a recursion relation for the coefficients $c_n$.

Numerical calculations are done in double-precision FORTRAN. For a given set of parameters $c$, $\varepsilon$, and $\delta$ one calculates $\lambda$, and for a given $c$, all the $c_n$'s up to $n = N$. The constant $c_1$ is then determined numerically with a modified Newton-Raphson method by finding the root of the equation $\eta_\lambda(x = 0) = 0$.

We have tested the convergence of the solution by calculating $\eta(x = 0)$ for large values of $N$, for $c = 1.11$, $\varepsilon = 0.22$, and $\delta = 0.5$. The values of $\eta(x = 0)$ converge to within $3 \cdot 10^{-11}$ of each other for values of $N = 4000$ and $N = 5000$. The results for $N = 1000$, for three different sets of parameters are summarized in table I, with $\eta_k$ referring to the $\text{sech}^2$ solution of the KdV equation, with the same parameters $c$, $\varepsilon$, and $\delta$, hence a solution moving with the same dimensionless velocity. Solutions for different values of $\varepsilon$ and $\delta$ are simply obtained from the above by changing the vertical and $x$ scales.

In fig. 1 we plot the difference $\Delta(x)$ between the fifth-order KdV equation solution and the KdV equation solution as a function of $x$. As seen from fig. 1, the differences between the fifth-order KdV equation solution and the KdV equation solution are,

<table>
<thead>
<tr>
<th>Table I. - Maximal amplitudes.</th>
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