ALGORITHMS GENERATING ALTERNATIVE SOLUTIONS FOR A MULTICRITERION LINEAR PROGRAMMING MODEL

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Modern support systems for group decisions in situational centers assume the availability of collective discussion and decision-making subsystems. One of the main functions of such subsystems is to generate alternative decisions for the given topic, which are then ranked in a certain way or from which one decision is chosen by consensus. The efficiency of the system largely depends on the availability of a procedure that generates "reasonable" (nearly optimal) alternative decisions in real time.

The most widespread model that describes economic, military, and social process is the multicriterion linear programming model (MCLP):

\[
\begin{align*}
\text{find} & \quad \{ \max c_1(x), \max c_2(x), \ldots, \max c_k(x) \} \\
\text{subject to} & \quad Ax \leq b, \quad x \geq 0,
\end{align*}
\]

where \( c_i(x), i = 1, k \) are linear objective functions, \( A \) is an \( m \times n \) matrix, \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), \( b = (b_1, b_2, \ldots, b_m) \in \mathbb{R}^m \).

For a wide range of important applied problems, such as allocation of state orders and scarce resources, budget negotiations, determination of strategic stability, or preparation of military operations, Eqs. (1), (2) reduce to the form

\[
\begin{align*}
\text{find} & \quad (\max x_1, \max x_2, \ldots, \max x_n) \\
\text{subject to constraints} & \quad (2) \text{ with the additional condition that the elements of the matrix } A \text{ and the vector } b \text{ are nonnegative.}
\end{align*}
\]

The objective is naturally set by the human decision maker in the category of decision criteria. Thus, the decision maker specifies the vector

\[
x^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})
\]

If constraints (2) are satisfied for the vector \( x^{(0)} \), then there is no problem: the objective can be met. Otherwise, we have a choice problem, because the initial objective cannot be achieved subject to the given constraints (2). The difficulty can be resolved in two ways:

1) first, we can alter (relax) the objective so that it remains optimally close (in a given sense) to the original objective \( x^{(0)} \) and yet satisfies the given constraints;

2) second, we can alter the constraints, i.e., the elements of the matrix \( A \) (the technologies) or (and) the components of the right-hand side vector \( b \) (the resources) so as to achieve the sought objective.

Problems of the first class constitute the traditional "concentrated" MCLP problem [1], whereas problems of the second class are classified as system optimization problems [2, 3].

Let us consider efficient interactive procedures that generate alternative solutions for these classes of problems.
We make a change of variables, introducing relative values of the unknowns, \( x_i = \alpha x_i^{(0)}, 0 \leq \alpha_i \leq 1, i = 1, n \). Equations (2), (3) thus reduce to the form:

\[
\text{find } \{\max \alpha_1, \max \alpha_2, \ldots, \max \alpha_n\} \tag{4}
\]

subject to

\[
\sum_{i=1}^{n} a_i^j x_i^{(0)} \leq b_j, 0 \leq \alpha_i < 1, j = 1, m. \tag{5}
\]

Denote by \( I \) the index set \( (1, 2, \ldots, n) \), by \( J \) the index set \( (1, 2, \ldots, m) \), by \( I_j \) the subset of indices \( i, i \in I \), of constraint \( j \) in Eq. (5) for which \( a_i^j \neq 0 \).

Equations (4), (5) have a whole set of Pareto-optimal (efficient) solutions.

Solving Eqs. (4), (5) with the aid of supplementary constraints, we obtain the set of alternative solutions.

The traditional methods [1] of ranking or scalarizing the criteria are ineffective, because they allow the decision maker to express the attitude toward objectives only on a qualitative level, with very limited quantitative implications.

The algorithms proposed here for the class of MCLP problems remove this shortcoming and, most importantly, are applicable to large problems. The algorithm finds one Pareto-optimal solution in time \( O(m \times n) \), compared with \( O(n^m) \) for the simplex method, i.e., the proposed algorithm is computationally optimal.

**GOAL-DIRECTED GENERATION OF ALTERNATIVE SOLUTIONS**

1. Determination of an achievable goal \( \tilde{x}_{(1)} \) (i.e., a goal that satisfies the constraints (5)) maximally close to \( x^{(0)} \) by all components. This is the traditional MCLP problem with equally important goal components [1] and is solvable analytically:

\[
a_{1h}^{(1)} = \min_{j \in J} \frac{b_j}{\sum_{i \in I} a_i^j x_i^{(0)}}. \tag{6}
\]

Given the constraints \( 0 \leq \alpha_i \leq 1 \), we finally obtain

\[
\tilde{a}_{1h}^{(1)} = \begin{cases} a_{1h}^{(1)}, & \text{if } a_{1h}^{(1)} \leq 1, \\ 1, & \text{if } a_{1h}^{(1)} > 1. \end{cases} \tag{6'}
\]

If the set \( I_1 \neq I \), then

\[
a_{12}^{(1)} = \min_{j \in J_1 \setminus I_1} \frac{b_j - \tilde{a}_{1I_j}^{(1)} \sum_{i \in I_j \setminus I_1} a_i^j x_i^{(0)}}{\sum_{i \in I_j \setminus I_1} a_i^j x_i^{(0)}}. \tag{7}
\]

\[
\tilde{a}_{12}^{(1)} = \begin{cases} a_{12}^{(1)}, & \text{if } a_{12}^{(1)} \leq 1, \\ 1, & \text{if } a_{12}^{(1)} > 1. \end{cases} \tag{7'}
\]

and so on, until we have determined the entire set of \( \tilde{a}_i, i \in I \), i.e., \( \bigcup_s I_f = I \).