Two-Sided Bounds Uniform in the Real Argument and the Index for Modified Bessel Functions

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ABSTRACT. Bounds uniform in the real argument and the index for the functions $a_\nu(x) = xI'_\nu(x)/I_\nu(x)$ and $b_\nu(x) = xK'_\nu(x)/K_\nu(x)$, as well as for the modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$, are established in the quadrant $z > 0, \; \nu \geq 0$, except for some neighborhoods of the point $z = 0, \; \nu = 0$.

KEY WORDS: two-sided bounds, Bessel function, modified Bessel functions, upper and lower barriers, the differential inequality theorem.

Introduction

The present paper was written in the process of justifying rapidly converging iteration methods with splitting boundary conditions for solving the first boundary value problem for Stokes type systems singularly perturbed in circularly symmetric domains (e.g., disk, annulus, ball, spherical layer). Such methods were proposed in [1-3] and studied there for a domain that is a layer between two hyperplanes under the assumption of periodicity along orthogonal directions parallel to the boundary hyperplanes. In [4] a similar iteration method with splitting boundary conditions was proposed and studied for a singular equation of biharmonic type in the case of a ball in $\mathbb{R}^n$.

To justify these methods for a domain that is an annulus or a spherical layer, we need to find bounds uniform in the real argument $x$ and the index $\nu$ for the functions

$$a_\nu(x) = \frac{xI'_\nu(x)}{I_\nu(x)} \quad \text{and} \quad b_\nu(x) = \frac{xK'_\nu(x)}{K_\nu(x)}$$

(1)

on the domains $D_\tau = \{(x, \nu) : x > 0, \; \nu \geq 0, \; \sqrt{x^2 + \nu^2} \geq \tau_*, \; \tau_* > 0\}$, where $I_\nu(x)$ and $K_\nu(x)$ are modified Bessel functions of the first and the third kind, respectively (the latter are Macdonald functions), as well as uniform bounds for the functions $I_\nu(x)$ and $K_\nu(x)$ themselves on the same manifolds. These functions form a fundamental system of solutions of the differential equation

$$x^2u'' + xu' - (x^2 + \nu^2)u = 0, \; x > 0.$$  

(2)

Moreover, $I_\nu(x)$ is a solution of Eq. (2), which is bounded at zero and exhibits the asymptotic behavior

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}}(1 + O(x^{-1})), \quad x \to \infty,$$

(3)

and $K_\nu(x)$ is a solution of Eq. (2) which exponentially decreases with increasing $x$ and exhibits the asymptotic behavior

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}}e^{-x}(1 + O(x^{-1})), \quad x \to \infty$$

(4)

(e.g., see [5, 6]).

The literature on the asymptotic behavior of Bessel functions, including modified Bessel functions, is quite extensive. For the functions $I_\nu(x)$ and $K_\nu(x)$, as well as for their derivatives, asymptotic expansions for large values of the argument and bounded values of the index, uniform asymptotic expansions for large

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values of the index, and several other expansions are well known. However, in standard textbooks and papers concerned with Bessel functions (e.g., [5-9]), we could not find any two-sided bounds uniform in \( x \) and \( \nu \) (necessary for our goals) for the functions \( I_\nu(x) \) and \( K_\nu(x) \), as well as for the functions (1).

Nevertheless, the required estimates for the functions (1) can be obtained by using a simple technique based on the differential inequality theorem (e.g., see [10]) and on the construction of refined upper and lower barriers defined below (and also essentially using the power expansion of the function \( a_\nu(x) \) at zero up to \( O(x^6) \) and the asymptotics of the function \( b_\nu(x) \) as \( x \to \infty \) up to \( O(x^{-3}) \), but only for some chosen values of \( \nu \)). Note that this approach was used in [4] to obtain a rough upper bound for the function \( a_\nu(x) \). As to the required two-sided uniform bounds for the functions \( I_\nu(x) \) and \( K_\nu(x) \), they can then be obtained by direct integration of refined estimates for the functions (1) with regard to the normalizations (3) and (4). In the present paper we implement this plan and find bounds for the functions (1), \( I_\nu(x) \), and \( K_\nu(x) \), which we need to accomplish the goals discussed above.

§1. Basic estimates

Here we state the basic estimates for the functions (1), which we derive later in §3, and the basic estimates for the functions \( I_\nu(x) \) and \( K_\nu(x) \), which we justify in §4.

**Theorem 1.** The following two-sided bounds hold for the functions (1). For \( a_\nu(x) \):

\[
-\frac{x^2}{2(x^2 + \nu^2)^{3/2}} < a_\nu(x) - \frac{x^2}{2(x^2 + \nu^2)} < \frac{x^2}{2(x^2 + \nu^2)^{3/2}},
\]

and the upper bound holds for all \( x > 0 \) and \( \nu \geq 0 \), while the lower bound holds on the set

\[
\{ (x, \nu) : x > 0, \nu \geq \frac{1}{2} \} \cup \{ (x, \nu) : x > 0, \nu \geq 0, \sqrt{x^2 + \nu^2} \geq \frac{\sqrt{7} + 2}{3} \}.
\]

For \( b_\nu(x) \):

\[
-\frac{\beta x^2}{(x^2 + \nu^2)^{3/2}} < b_\nu(x) + \frac{x^2}{2(x^2 + \nu^2)} < \frac{x^2}{8(x^2 + \nu^2)^{3/2}},
\]

where \( \beta > 1/2 \) is arbitrary, and upper bound holds on the set

\[
\{ (x, \nu) : x > 0, \nu \geq 0, \sqrt{x^2 + \nu^2} \geq \frac{1}{5} \},
\]

while the lower bound holds on the set

\[
\{ (x, \nu) : x > 0, \nu \geq 0, \sqrt{x^2 + \nu^2} \geq \frac{2\beta}{(2\beta - 1)} \}.
\]

**Theorem 2.** The following two-sided bounds hold for modified Bessel functions. For \( I_\nu(x) \):

\[
e^{-\frac{1}{2\sqrt{x^2 + \nu^2}}} \leq I_\nu(x) \sqrt{\frac{\nu}{2\pi}}(x^2 + \nu^2)^{1/4} e^{-\left(\sqrt{x^2 + \nu^2} + \nu \ln \frac{x}{\nu + \sqrt{x^2 + \nu^2}}\right)} \leq e^{-\frac{1}{2\sqrt{x^2 + \nu^2}}},
\]

and the lower bound holds for all \( x > 0 \) and \( \nu \geq 0 \), while the upper bound holds on the set (6). For \( K_\nu(x) \):

\[
e^{-\frac{1}{2\sqrt{x^2 + \nu^2}}} \leq K_\nu(x) \sqrt{\frac{\nu}{2}}(x^2 + \nu^2)^{1/4} e^{\left(\sqrt{x^2 + \nu^2} + \nu \ln \frac{x}{\nu + \sqrt{x^2 + \nu^2}}\right)} \leq e^{-\frac{6}{\sqrt{x^2 + \nu^2}}},
\]

where \( \beta > 1/2 \) is arbitrary, and the lower bound holds on the set (8), while the upper bound holds on the set (9).