Partitions of the Phase Space of a Measure Preserving $\mathbb{Z}^d$-Action Into Towers

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Abstract. Alpern proved that the phase space of an aperiodic measure preserving automorphism $T$ can be decomposed into Rokhlin-Halmos towers of given heights $h_i$ and weights $m_i$ whenever the numbers $h_i$ are relatively prime. In this paper an extension of the Alpern theorem to the case of free $\mathbb{Z}^d$-actions is given. Namely, we investigate the decomposability of the action phase space into towers of rectangular form and present conditions on the configuration (the set of tower forms) sufficient for the existence of such a decomposition. The proof of the main result uses the technique of filings.

Key words: probability space, Rokhlin-Halmos towers, aperiodic measure preserving automorphisms, free action of the group $\mathbb{Z}^n$, rectangular tilings, ergodic theory.

§1. Introduction. The Rokhlin-Halmos lemma and its variations

Let us consider a probability space $X$ isomorphic to the interval $[0, 1]$ with the Lebesgue measure $\mu$. The triple $(X, \mathcal{B}, \mu)$, where $\mathcal{B}$ is the $\sigma$-algebra of measurable sets, is called the Lebesgue space. An automorphism of the space $(X, \mathcal{B}, \mu)$ is defined as an invertible measurable transformation $T: X \rightarrow X$ such that $\mu(T^{-1}A) = \mu(TA) = \mu(A)$ for any $A \in \mathcal{B}$. An automorphism $T$ is called aperiodic if $T^{a_1}z = T^{a_2}z \Rightarrow a_1 = a_2$ for almost all points $z \in X$.

The following well-known Rokhlin-Halmos lemma (see [1]) has numerous applications in ergodic theory. Among them, applications to approximation theory (see [1, 2]) and entropy theory (see [3]) can be mentioned. A number of other applications are discussed in what follows.

Lemma 1. If $T$ is an aperiodic automorphism of the Lebesgue space $(X, \mathcal{B}, \mu)$, then for any $h \in \mathbb{N}$ and $\varepsilon > 0$ there exists a Rokhlin-Halmos tower consisting of disjoint sets

$$B, TB, T^2B, \ldots, T^{h-1}B \in \mathcal{B}, \quad \text{such that } \mu\left(\bigcup_{j=0}^{h-1} T^jB\right) > 1 - \varepsilon.$$

The set $B$ and the number $h$ will be called the base and the height of the tower, respectively. Note that a similar statement with $\varepsilon = 0$ is not true in general, e.g., for completely ergodic automorphisms. (An automorphism $T$ is called completely ergodic if any power $T^n$, $n \neq 0$, of $T$ is ergodic.)

The following statement, proved by Lehrer and Weiss in [4], is a variation of Lemma 1.

Proposition 1. Let $T$ be a completely ergodic nonsingular map. Then for any $h \in \mathbb{N}$ and $E \subseteq X$, $\mu(E) > 0$, there exists a tower $\bigcup_{j=0}^{h-1} T^jB$ such that

$$\bigcup_{j=0}^{h-1} T^jB \supseteq X \setminus E.$$

In other words, for a completely ergodic map, the difference $X \setminus \bigcup_{j=0}^{h-1} T^jB$ can be placed into an arbitrary preassigned set $E$ of positive measure. Recall that the Rokhlin-Halmos lemma only tells that the measure of this difference can be made arbitrarily small.

Note that for $\mathbb{Z}^2$-actions an assertion similar to the Lehrer-Weiss lemma does not hold (see [5]). In particular, the following fact is true:
Proposition 2. Suppose that commuting transformations $T$ and $S$ of the space $(X, \mathcal{B}, \mu)$, positive integers $m \geq 2$ and $n \geq 3$, and a set $U$ (tower of size $m \times n$) satisfy the relations:

$$TU \cup U = U \cup SU = X, \quad U = \bigcup_{a=0}^{m-1} \bigcup_{b=0}^{n-1} T^aS^bC.$$

If the $\mathbb{Z}^2$-action generated by the automorphisms $T$ and $S$ is completely ergodic, then the entropy of this action with respect to the partition $\xi = \{U, X \setminus U\}$ is 0. (Thus the action in question has a nontrivial factor with zero entropy.)

The following sharpening of the Rohlin–Halmos lemma was used by Thouvenot in [3].

Proposition 3. Suppose that $P$ is a finite partition of the space $(X, \mathcal{B}, \mu)$, $h \in \mathbb{N}$, and $\varepsilon > 0$. There exists a Rohlin–Halmos tower of height $h$ with base $B$ that satisfies the assumptions of Lemma 1 and has the following property:

$$\forall A \in P \quad \mu(A \cap B) = \mu(A) \cdot \varepsilon.$$

The main point of this proposition is in the fact that we can choose the set $B$ to be “independent” of the partition $P$.

Now we proceed to the formulation of the Alpern theorem (Theorem 1), which we intend to generalize in this paper.

Definition 1. Any set $P \subseteq \mathbb{N}$ is called a (one-dimensional) configuration. A sequence $m = \{m_h\}_{h \in P}$ of positive numbers such that $\|m\|_1 := \sum_{h \in P} m_h = 1$ is called a $P$-distribution.

Consider an aperiodic automorphism $T$ of the Lebesgue space $(X, \mathcal{B}, \mu)$ and a pair $(P, m)$, where $m$ is a $P$-distribution.

Theorem 1. Let $\gcd(P) = 1$. Then there exist measurable sets $\{B_h\}_{h \in P}$ such that

$$X = \bigcup_{h \in P} B_h^{[0,\ldots,h)} = \bigcup_{h \in P} \left( \bigcup_{j=0}^{h-1} T^jB_h \right), \quad \mu(B_h^{[0,\ldots,h)}) = m_h, \quad h \in P,$$

where $[0,\ldots,h) := \{j \in \mathbb{Z} : 0 \leq j < h\}$ is an integer interval of length $h$.

This theorem was proved by Alpern in [6] and [7]. (A new proof of this result can be found in [8]; see also [9].)

Alpern applied Theorem 1 to solve problems connected with the approximation of a transformation $T$ of the Lebesgue space by automorphisms conjugate to a given transformation $S$. In particular, he proved the following two statements.

(1) Let $T$ and $S$ be two automorphisms of $X$, $S$ being aperiodic. Let $\mathfrak{A}$ be a finite subalgebra of $\mathcal{B}$ such that none of the sets $A \in \mathfrak{A}$ has the property $\exists n$ $T^nA = A$. Then there exists an automorphism $\widehat{S} = \theta^{-1}S\theta$ such that $\widehat{S}A = TA$ for all $A \in \mathfrak{A}$.

(2) Let $T$ and $S$ be automorphisms of $X$ such that $S$ is aperiodic and $T$ is aperiodic on $F \in \mathcal{B}$, $\mu(F) < 1$ (i.e., $F$ does not contain periodic points). Then, if either (a) $T$ is ergodic and $\mu(F \cup TF) < 1$ or (b) $T$ is weakly mixing, then there exists an automorphism $\widehat{S} = \theta^{-1}S\theta$ such that $\widehat{S}x = Tx$ for $\mu$-almost all $x \in F$.

Alpern used the above statements to prove that the ergodicity property is typical in the class of Lebesgue measure preserving homeomorphisms of $\mathbb{R}^d$ (see [10]).

A particular case of the Alpern theorem with $P = \{p, p+1\}$ was used by Ryzhikov in [11] in the proof of the following statement.