Extremum Problems for Golubev Sums

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ABSTRACT. Suppose that $G$ is a finitely connected domain with rectifiable boundary $\gamma$, $\infty \in G$, the domains $D_1, \ldots, D_s$ are the complements of $G$, the subsets $F_j \subseteq D_j$ are infinite and compact, $n_j \geq 1$, $j = 1, \ldots, s$, are integers, $\lambda_0$ is a complex-valued measure on $\gamma$, and

$$\omega_j(t) = \int_{\gamma} (t - \xi)^{-n_j} d\lambda_0, \quad t \in F_j, \quad j = 1, \ldots, s.$$  

We consider the extremum problem

$$\beta = \sup_{\mu_1, \ldots, \mu_s} \left| \sum_{j=1}^{s} \int_{F_j} \omega_j(t) d\mu_j \right|,$$

where $\mu_j$, $j = 1, \ldots, s$, are complex-valued measures on $F_j$ and

$$\left| \sum_{j=1}^{s} \int_{F_j} (t - z)^{-n_j} d\mu_j \right| \leq 1, \quad z \in G,$$

are Golubev sums. We prove that $\beta = \Delta$, where

$$\Delta = \inf \int_{\gamma} |d\lambda| \left/ \int_{\gamma} (t - \xi)^{-n_j} d\lambda = \int_{\gamma} (t - \xi)^{-n_j} d\lambda_0 = \omega_j(t), \quad t \in F_j, \quad j = 1, \ldots, s.$$  

We also establish several other relations between these and other extremal variables.

KEY WORDS: extremum problem, analytic function, rectifiable curve, compact set, Cauchy potential, Golubev sum, Laurent expansion, complex-valued measure.

In the present paper we study extremum problems for bounded analytic functions representable in terms of a special analytic apparatus, which is called Golubev sums. It seems a priori that this apparatus has an analytic advantage over the traditional apparatus of Cauchy type integrals (the "Cauchy potentials"), which are a specific case of Golubev sums. The term "Golubev sums" is justified by the fact that similar analytic tools were first studied in Golubev's thesis [1, 2] defended in 1916. This thesis was a noticeable milestone in the history of the theory of analytic functions. Let $F$ be a rectifiable curve. Golubev studied the possibility of representing a function $f(z)$ analytic outside $F$ and satisfying the condition $f(\infty) = 0$ by the series

$$f(z) = \sum_{k=1}^{\infty} \int_{F} \frac{\varphi_k(t)}{(t - z)^k} dt,$$

where $\varphi_k(t)$ are functions integrable on $F$ with respect to the measure $|dt|$ and the series (1) converges in the interior of the complement of $F$. Together with the infinite series (1), Golubev also studied representations by finite sums of the form (1) by treating them, in a certain sense, as Laurent expansions of functions with finite-order singularities (poles).

Golubev asked if any function $f(z)$ analytic outside $F$ and satisfying the condition $f(\infty) = 0$ can be represented by the series (1). This problem was studied by Khavin [3-5] and Vitushkin [6]. Khavin
obtained the affirmative answer to Golubev's question. Moreover, this answer was obtained in a more general situation, where $F$ is a connected compact set (a continuum), the integration in (1) is carried out with respect to a measure $d\mu \geq 0$ given on $F$ and satisfying a single assumption (namely, the measure must be "essential" in the sense that if $e \subset F$ and $\mu(e) = 0$, then $F \setminus e = F$, where $\overline{A}$ is the closure of the set $A$), the functions $\varphi_k(t)$ are square-integrable on $F$, and their norms decrease fast enough to ensure a "good" convergence of the series (1). Originally, Khavin [3] stated his theorem for an arbitrary compact set $F$ (without the connectivity assumption). However, this statement turned out to be false. Vitushkin [6] constructed an everywhere disconnected compact set $F$ and a function $f(z)$ continuous in (the Riemann sphere) $\mathbb{C}$ and analytic outside $F$. This function cannot be written as a Cauchy potential over $F$ and cannot even be presented by Golubev series with arbitrary complex-valued measures $d\mu_k$ (instead of $\varphi_k(t) dt$) in its terms. Val'skii [7] proved the following statement: if an everywhere discontinuous compact set $F$ is sufficiently large, i.e., of positive Newtonian volume, then there always exists a function $f(z)$ continuous in the entire plane and analytic outside $F$, which cannot be presented as a Cauchy potential. The author of the present paper noted [8] that this function also cannot be represented as a Golubev series (again with arbitrary complex-valued measures). The result obtained by Vostretsov [9; 10, Chap. II, §9] can also be regarded as related to these problems. Vostretsov showed that any function $f(z)$ analytic in the unit disk can be written in the form

$$f(z) = \int_{\mathcal{T}} g\left(\frac{t}{t-z}\right) \varphi(t) dt,$$

where $g(u)$ is an entire function of a certain growth rate and $\varphi(t)$ is a complex-valued function square-integrable on the unit circle $\mathcal{T}$. Generally speaking, $g$ and $\varphi$ are different for each $f(z)$. If we compare this representation with the series (1), then we see that here $\varphi_k(t) = C_k t^k \varphi(t)$, where $C_k$ are constants. Starting from the Khavin method, Aizenberg [11] generalized the Vostretsov representation to finitely connected domains with rectifiable boundaries. The paper [3] also contains a criterion for the representation of an analytic function by Golubev series, described by Aizenberg, to consist only of finitely many terms, i.e., for a function to be representable as a "Golubev sum."

The present paper originates the study of extremum problems for bounded analytic functions representable by Golubev sums. Here we modify the construction of the sums by assuming that the measures representing sums with different derivatives of the Cauchy kernel are concentrated on different compact sets. We consider the simplest, in a sense, configuration possible here. This is justified by the fact that such a configuration requires the largest number of extremal variables, which are different in origin but turn out to be equal to each other. The goal of the paper consists precisely in finding such variables and proving that they are equal. The author hopes to continue this study for the most general configurations in the construction of sums and for extremal variables that are analogs of the analytic capacity for such configurations.

Now let us agree on the notation. Let $\mathcal{C}(F)$ be the space of complex-valued functions continuous on a compact set $F$. This space is equipped with the usual (uniform) norm. By $\mathcal{M}_A$ we denote the set of complex-valued finite regular Borel measures on the set $A \subset \mathbb{C}$ (where $\mathbb{C}$ is the complex plane and $\overline{C}$ is the Riemann sphere, i.e., the extended complex plane). In this paper we do not consider other measures. By $L_p(A, \nu)$, $p \geq 1$, we denote the Lebesgue space of complex-valued functions $\varphi(z)$ on the set $A$, for which $|\varphi(z)|^p$ are integrable with respect to a positive measure $d\nu$ given on $A$. By $|d\nu|$ we denote the measure defined as the total variation of the measure $d\nu$.

Suppose that $D_1, \ldots, D_s$, $s \geq 1$, are finite simply connected domains in $\mathbb{C}$ whose boundaries $\gamma_1, \ldots, \gamma_s$ are closed Jordan rectifiable curves and the closures $\overline{D}_1, \ldots, \overline{D}_s$ do not intersect pairwise. Further, we write

$$D = D_1 \cup \cdots \cup D_s, \quad G = \overline{\mathbb{C}} \setminus \overline{D}, \quad \gamma = \gamma_1 + \cdots + \gamma_s = \partial G.$$

By $E_1(G)$ we denote the Smirnov class of functions $\psi(z)$ that are analytical in $G$, satisfy the condition $\psi(\infty) = 0$, have boundary values integrable on $\gamma$ with respect to the measure $|d\xi|$, and can be represented as the following Cauchy integral in terms of these boundary values:

$$\psi(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{\psi(\xi) d\xi}{\xi - z}.$$