Singularity Properties
of the Bubble Diagrams of \( S \)-Matrix Theory.

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(ricevuto il 7 Marzo 1967)

Summary. A general discussion is given of singularity properties arising from the «contraction» of internal lines of the «bubble diagrams» of \( S \)-matrix theory, such as the terms of unitarity equations. The circumstances in which one may contract such internal lines are described and an argument is given for deciding what part of the resulting curve is actually singular. The relevance of these results to the derivation of singularity properties of scattering amplitudes from unitarity is discussed.

1. – Introduction.

This paper is concerned with the singularity structure of functions represented by «bubble diagrams», such as the one shown in Fig. 1. A bubble diagram is a set of circles called bubbles representing mass shell scattering amplitudes each evaluated on some specified sheet (not necessarily the physical sheet), connected by lines representing particles on their mass shell. To obtain the function corresponding to such a bubble diagram (which is a function of the momenta of the particles represented by the external lines of the diagram) one takes the product of the scattering amplitudes exhibited in the

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diagram and integrates over physical (mass shell) values of the momenta of the particles corresponding to the internal lines of the diagram. If one represents the singularities of the bubbles by Feynman diagrams, one sees that the following analysis applies also to the singularity structure of functions represented by Feynman-type diagrams, some of whose internal lines are mass shell delta-functions rather than propagators. Recalling the Cutkosky prescription (1) it is clear that discontinuities of Feynman diagrams are given by such functions.

The importance of bubble diagrams lies in their occurrence in unitarity equations. Several recent papers (2-5) have derived certain singularity properties of scattering amplitudes by making an analytic continuation of a unitarity equation from a region $R_1$ to a region $R_2$ past a singularity; because the terms in the unitarity equation may contain the singularity in question, the unitarity equation continued from $R_1$ to $R_2$ will be different from the unitarity equation operating in $R_2$. By comparing the two equations the singularity structure is found. This procedure obviously requires knowledge of which terms in the unitarity equation possess a particular singularity on a particular sheet.

The topic that this paper discusses is the possession, by bubble diagrams, of singularities corresponding to the contraction of internal lines of the diagram. CUTKOSKY (1) implies that when we consider the singularities of discontinuities of Feynman diagrams we are not allowed to omit any of the lines that have been put on the mass shell. However counter-examples to this proposition have been found (6,7).

In Sect. 2 we present the general theory which tells one whether it is permissible to contract internal lines of bubble diagrams, and, if so, what parts of the resulting curve are singular. We find that, in general, one may contract internal lines provided they form closed loops, in agreement with the results of ref. (6,7); the singular parts of the curve arising from a contracted diagram are bounded by the points of (tangential) contact with the curve from the uncontracted diagram (or by points of contact with curves from still further contracted diagrams, in the usual hierarchical way (8,9)). In Sect. 3 we