On the $S$-Matrix of Highly Absorptive Resonant Waves: 
The $\pi N$ $S_{31}$ Wave.

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Summary. — It is shown that the $S$-matrix of a highly absorptive resonant wave is dominated by a factor called outer factor. As an application an analytic expression for $\pi N$ $S$-matrix in $S_{31}$ wave is found.

In a recent paper (1) the requirement of causality was studied in the case of a relativistic finite-range interaction without bound states. Two different causality conditions were studied. In both cases the $S$-matrix can be factorized as

$$ S_i(w) = \exp \left[ -2ikR \right] S_i^x(w) S_i^y(w), $$

$w$ and $k$ being the energy and momentum of the incident particle in the c.m. system and $R$ the range of the interaction. Equation (1) holds in the upper half-plane of $w$. $S_i^x$ is a function analytic and bounded in the upper half-plane of $w$. It has the following property: if $F(w)$ is also analytic and bounded in the upper half-plane and

$$ |S_i^x| = |F|, $$

in the real axis, then

$$ |S_i^y| \geq |F| \text{ a.e.} $$

in the upper half-plane

Functions such as $S_i^y$ are called outer functions (see ref. (2) Chapt. 5 and 8).

$S^\pi_i$ has the form (1,2)

\[ S^\pi_i(w) = \exp \left[ \int_{-\infty}^{+\infty} \log |S^\pi_i(t)| \frac{wt + 1}{w - t} \left( \frac{1}{1 + t^2} \right) dt \right]. \]

$S'_i$ is called inner function (1,2) and has the form

\[ S'_i(w) = \exp [iaw] \prod \frac{w - \Omega_n - i\Gamma_n/2}{w - \Omega_n + i\Gamma_n/2}, \]

where $a > 0$; $\Gamma_n > 0$. The second factor is called Blaschke product (*). It is clear that $|S'_i| = 1$ along the real axis so that

\[ |S_i(w)| = |S^\pi_i(w)| \]

for real $w > m$. For this reason when a wave is not very absorptive the outer factor is close to unity and varies slowly. If this wave resonates it turns out that the behaviour of the $S$-matrix is mainly given by the inner factor. More precisely the dominant factor is

\[ \frac{w - \Omega - i\Gamma/2}{w - \Omega + i\Gamma/2} \]

for a resonance of energy $\Omega$ and width $\Gamma$. In ref. (3) all $\pi N^*$ resonances below 1 GeV with elasticity (**) higher than $\frac{1}{2}$ were studied neglecting the outer factor. The agreement with experiment was excellent (***)

In this paper we will show that when a resonant wave is very absorptive the situation is reversed and the outer part is much more important than the inner factor. The inner factor reduces to the exponential factor in (3) without the Blaschke product (all $I_n = 0$). We will consider first the case of an ideal resonance and after that the case of a real one: the $N^*(1688) T J^p = \frac{3}{2}^+$. We will see that the $S$-matrix can be expressed without the Blaschke product.

1. From (1) we can write

\[ \delta_i = i^\pi_i + i^I_i - kR, \]

(*) In fact $S'_i$ can also contain a factor called singular function. Since it plays no role in the following we neglect it.


(**) Elasticity of a resonance is defined as the ratio of the elastic-channel width to the total width.

(***) In ref. (1) a slightly different form of (1) was used. An extra inner function in the $k$ variable was present.