Electromagnetism and Gravitation.

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As a first step towards a proper unification of the two fundamental macroscopic fields, gravitation and electromagnetism, it is essential to understand a direct interaction of these two fields. In the framework of general relativity, the study of electromagnetism in curved space-time has so far been limited to the study of Einstein-Maxwell equations. In deriving the Einstein-Maxwell equations from an action principle, the interaction is considered mainly through the introduction of the $\sqrt{-g}$ term along with the electromagnetic action Lagrangian.

In our opinion, since the electromagnetic field energy should also produce curvature of the manifold, the combined field equations should be obtained by the introduction of a more explicit interaction term which makes use of the fundamental tensors of both these fields. Recently we have defined (1) a new invariant $S = R_{hijk}F^{hi}F^{jk}$ for electromagnetic fields in curved space-time wherein $R_{hijk}$ is the Riemann-Christoffel curvature tensor and $F_{ij}$ the electromagnetic field tensor, and further shown that this invariant plays the part of the Lorentz invariant for conformally flat space-times in the framework of Einstein-Maxwell equations.

We now start from a new action Lagrangian

\begin{equation}
\mathcal{L} = R \sqrt{-g} + \alpha E \sqrt{-g} + \beta S \sqrt{-g}
\end{equation}

in the four-dimensional riemannian manifold:

\begin{equation}
\text{d}s^2 = g_{ij} \text{d}x^i \text{d}x^j,
\end{equation}

wherein

\begin{equation}
R = g^{ij}g^{hk}R_{hijk}, \quad E = F^{ij}F_{ij},
\end{equation}

$\alpha$ and $\beta$ being the coupling constants. The electromagnetic field tensor $F_{ij}$ is defined in the usual way:

\begin{equation}
F_{ij} = (\varphi_{i,j} - \varphi_{j,i}),
\end{equation}

where $\varphi_i$ is the four-vector potential, and thus the Lagrangian is gauge invariant. Applying the method of variations, we have

\[ \delta \int \mathcal{L} \, dx = 0 \]

with $g_{ij}$ and $\varphi_i$ being the independent field variables whose variations vanish at the boundary. We get the set of equations:

\begin{align*}
\frac{\partial \mathcal{L}}{\partial g_{ij}} &= 0 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \varphi_i} = 0 ,
\end{align*}

i.e.

\[ R_{ij} - \frac{1}{2} R g_{ij} + 2 \alpha E_{ij} + 4 \beta (S_{ij} - \frac{1}{2} g_{ij} S + \frac{1}{2} P_{ij}) = 0 \]

and

\[ \frac{\partial}{\partial x^i} (\alpha \sqrt{-g} F^{ij} + \beta \sqrt{-g} K^{ij}) = 0 . \]

Here $R_{ij}$ is the Ricci tensor, $R$ the scalar of curvature, $E_{ij}$ the electromagnetic energy-momentum tensor,

\begin{align*}
P_{ij} &= (F^i_h F^k_j)_{;ih} - T_{i;k} F^{hl} F^j_h , \\
S_{ij} &= R_{i;k_l} F^{p_q} F^r_j , \quad \text{and} \quad K^{ij} = R^{i;jk} F_{pq} .
\end{align*}

Equations (7) and (8) constitute the new set of field equations for the combined electromagnetic and gravitational field. Taking the trace of (7) we find that

\[ R = 2 \beta [ S + (F^{hl} F^i_l)_{;ih} - R_{i;k} F^{hl} F^k_i] . \]

As a direct consequence of the above field equations one can see that if we start from a Riemannian space (2) of constant curvature

\[ R_{ijkl} = \lambda (g_{kl} g_{ij} - g_{ij} g_{kl}) \]

then we have

\[ S_{ij} = 2 \alpha F_{ri} F^{r_i} \quad \text{and} \quad R_{i;k} F^{hl} F^k_i = \lambda F_{ri} F^{r_i} . \]

Hence from the field equations we get

\[ 2 \alpha (F_{ri} F^{r_i} - \frac{1}{4} g_{ij} F_{kl} F^{kl}) + \beta [6 \lambda F_{ri} F^{r_i} - \lambda g_{ij} F_{kl} F^{kl} + 2 (F^{hl} F^i_l)_{;ih}] = - 3 \beta \lambda g_{ij} \]

and

\[ \frac{\partial}{\partial x^i} \left[ (\alpha + 2 \beta \lambda) \sqrt{-g} F^{ij} \right] = 0 . \]